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An Operational Calculus for the Euclidean Motion Group with Applications in Robotics and Polymer Science

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ABSTRACT. *In this article we develop analytical and computational tools arising from harmonic analysis on the motion group of three-dimensional Euclidean space. We demonstrate these tools in the context of applications in robotics and polymer science. To this end, we review the theory of unitary representations of the motion group of three dimensional Euclidean space. The matrix elements of the irreducible unitary representations are calculated and the Fourier transform of functions on the motion group is defined. New symmetry and operational properties of the Fourier transform are derived. A technique for the solution of convolution equations arising in robotics is presented and the corresponding regularized problem is solved explicitly for particular functions. A partial differential equation from polymer science is shown to be solvable using the operational properties of the Euclidean-group Fourier transform.*

1. Introduction

The Euclidean motion group, $SE(N)$,[1] is the semidirect product[2] of $\mathbb{R}^N$ with the special orthogonal group, $SO(N)$. That is, $SE(3) = \mathbb{R}^3 \triangleright SO(3)$. We denote elements of $SE(N)$ as $g = (\mathbf{a}, A) \in SE(N)$ where $A \in SO(N)$ and $\mathbf{a} \in \mathbb{R}^N$. The group law is written as $g_1 \circ g_2 = (\mathbf{a}_1 + A_1\mathbf{a}_2, A_1A_2)$, and $g^{-1} = (-A^T\mathbf{a}, A^T)$. Alternately, one may represent any element of $SE(N)$ as an $(N+1) \times (N+1)$ homogeneous transformation matrix of the form:

$$H(g) = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{0}^T & 1 \end{pmatrix}.$$

Clearly, $H(g_1)H(g_2) = H(g_1 \circ g_2)$ and $H(g^{-1}) = H^{-1}(g)$, and the mapping $g \to H(g)$ is an isomorphism between $SE(N)$ and the set of homogeneous transformation matrices.

Keywords and Phrases. convolution, Euclidean group, rigid body motion, diffusion equations, inverse problems, regularization, Fourier transform, harmonic analysis.

[1] The notation $SE(N)$ comes from the terminology Special Euclidean group of N dimensional space.

[2] The notation $\triangleright$ is used to denote semidirect product, as in [17]. We note that $\ltimes$ and $\rtimes$ are more commonly used.

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The motion group plays a central role in the kinematic geometry of mechanisms [1, 20, 3], robots [27, 30, 31], and machines [18]. $SE(3)$ and related groups are also important in computer vision and image processing [24, 19, 29]. In the past 40 years, the representation theory and harmonic analysis for the Euclidean group has been developed in the pure mathematics and mathematical physics literature. The study of matrix elements of irreducible unitary representation of $SE(3)$ was initiated by N. Vilenkin [34] in 1957 (some particular matrix elements are also given in [35]). The most complete study of $\widetilde{SE}(3)$ (the universal covering group of $SE(3)$) with applications to harmonic analysis was given by W. Miller in [28]. The representations of $SE(3)$ were also studied in [33, 32, 21, 17].

However, despite considerable progress in the representation theory of $SE(3)$ and other noncompact noncommutative groups, these achievements have not yet been widely incorporated in engineering and applied fields. In this article we try to fill this gap. We review the representation theory of $SE(3)$, derive the matrix elements of the irreducible unitary representations and define the Fourier transform for $SE(3)$. We derive new symmetry and operational properties of the Fourier transform and give explicit examples of Fourier transforms of functions on the motion group. We apply noncommutative harmonic analysis to two problems: (1) the solution of equations of the form

$$(f_1 * f_2)(g) = f_3(g)\,, \tag{1.1}$$

where $*$ denotes the convolution of functions on $SE(3)$, $f_1(g)$ and $f_3(g)$ are known functions, and $f_2(g)$ is to be determined; and (2) solutions to partial differential equations of the form

$$\frac{\partial f}{\partial L} = Df \tag{1.2}$$

that arise in polymer science where $f(g; L)$ is a probability density function on $SE(3)$ for each value of arclength L and D is an operator explained in Section 2.2.

Techniques for solving (1.1) where $g \in SE(2)$ were presented in [5]. We now consider the more complicated case of when $g \in SE(3)$. Because problem (1.1) is ill-posed, we must seek regularization techniques for performing approximate deconvolution. The approach taken here is to generalize Tikhonov regularization (see e.g., [16]) to the Euclidean group. Our approach to the solution of (1.2) is analogous to the way in which linear diffusion equations with constant coefficients are solved on the line. In order to solve both of these problems, two analytical tools are required: (1) the appropriate concept of Fourier transform; and (2) an understanding of how differential operators acting on functions on the group transform to algebraic operations on the Fourier transform. This is what we refer to as the *operational calculus.*

The remainder of this article is structured as follows. In Section 2 we illustrate a physical situation in which equations of the form of (1.1) and (1.2) appear. Section 3 reviews the representation theory of $SE(3)$. Sections 4 and 5 define the Fourier transform for $SE(3)$ and derive operational and symmetry properties of the Fourier transform of real-valued functions on $SE(3)$. In Section 6, explicit examples of Fourier transforms on $SE(3)$ are given. In Sections 7 and 8, operational properties derived in this article are used to regularize the solution of (1.1), and provide analytical solutions to (1.2), respectively.

2. Motivational Examples

Here we present in greater detail problems that motivate the need for an operational calculus for the Euclidean motion group.

2.1 Workspace Densities of Robotic Manipulators

In this subsection we explain why the inverse problem stated in (1.1) arises in the field of robotics.

A robotic manipulator arm is a device that is used to position and orient objects in the plane or in three-dimensional space. A manipulator is generally constructed of rigid links and actuators, such as motors or hydraulic cylinders, which cause all motion of the arm. If the actuators have only a finite number of states, as is the case with stepper motors or pneumatic cylinders, then the arm has a finite number of configurations and only a finite number of frames[3] are reachable by the hand. This is illustrated in Figure 1 for a manipulator composed of passive hinge-like joints and linear/translational actuators with binary states denoted '0' and '1'. This manipulator is capable of only reaching eight positions and orientations in the plane. Such a manipulator is called a binary manipulator [4]. The set of all reachable positions and orientations is called the *workspace*. Clearly, in the case depicted in Figure 1, the workspace is a discrete subset of $SE(2)$.

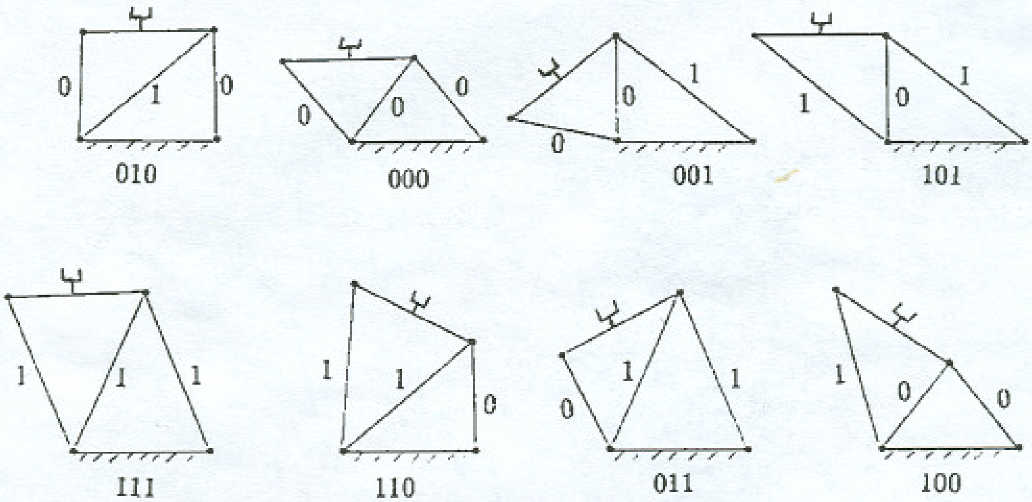

FIGURE 1 An 8-state binary manipulator.

For discretely actuated manipulators the *density* of reachable frames in $SE(N)$ determines how accurately a random position and orientation can be reached. This density information is also extremely important in planning the motions of discretely actuated manipulator arms [9]. Density is calculated directly by dividing a compact subset of $SE(N)$ containing the workspace into finite but small volume elements. The number of positions and orientations reachable by the end of the manipulator which lie in each volume element is stored. Dividing this number by the volume element size gives the average density in each element. Efficient methods for the calculation of this density histogram in the planar case are given in [10, 6]. A smooth density function can be used to approximate the shape of this density histogram as in [11]. Note that the density function always takes non-negative real values.

It is an important aspect of the manipulator design problem to specify the density of reachable frames throughout the workspace. That is, areas which must be reached with great accuracy should have high density, and those areas of the workspace which are less important need less density.

[3] A frame of reference in space is completely determined by the position of its origin and its orientation relative to another frame.

For relatively few actuators, the design problem may be solved by enumerating reachable frames (positions and orientations) and using an iterative procedure as discussed in [4]. However, to compute this workspace density function using brute force and iterating is computationally intractable for large n, e.g., it requires K^n evaluations of the kinematic equations relating actuator state to the resulting end frame for a manipulator with n actuators each with K states. In Figure 1, $K = 2$ and $n = 3$ so the problem is simple.

A grayscale of the density of frames reachable by a discretely actuated manipulator is shown in Figure 2 with several configurations of the arm superimposed.[4] This manipulator is essentially a serial cascade of modules with the same kinematic structure as in Figure 1, only now each leg has four states instead of two. Since each leg has four states (and thus the whole manipulator has $4^{30} \approx 10^{18}$ states) the workspace density cannot simply be computed using brute force. In fact, it would take years using current computer technology to enumerate all the positions and orientations of the frame attached to the end of the manipulator for each discrete configuration.

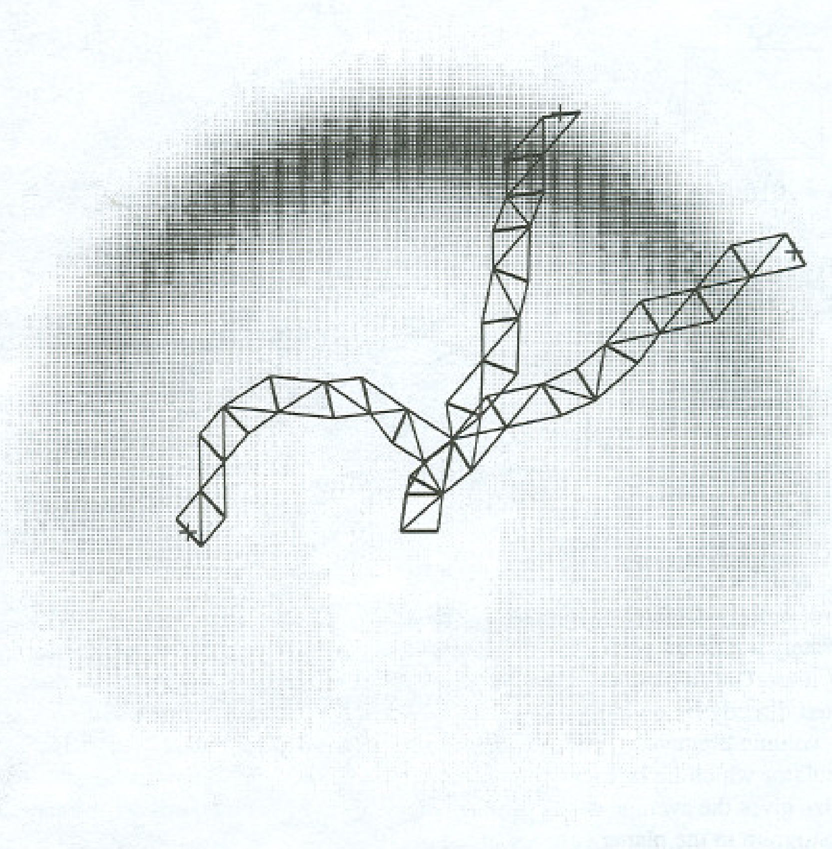

FIGURE 2 A discretely actuated manipulator with 4^{30} states.

The concept of convolution of real-valued functions on $SE(N)$ provides a powerful computational tool for computing this density efficiently [10, 11]. If we imagine that the manipulator is divided into two connected parts, then a density function $f_1(g)$ can be associated with those frames

[4]A density function on $SE(2)$ can be written as $f(g(x, y, \theta))$. What is shown is really the integral of this over θ (the orientation variable) from $-\pi$ to π so that a planar picture results.

reachable by the end of the lower half of the manipulator, and a density function $f_2(g)$ can be associated with the end of the upper half of the manipulator. f_1 is defined relative to the base frame, and f_2 treats the frame at the end of the lower segment as the base frame. That is, $f_1(g) = f_2(g)$ when the manipulator is cut into two equal parts and there are an even number of identical modules. However, f_1 and f_2 will not be the same function in more general scenarios. By adjusting kinematic parameters such that actuator strokes are limited or extended, the set of reachable frames (and thus the density) is altered. This is achieved mechanically by simply inserting or removing rigid stoppers that specify the physical actuator length corresponding to the discrete states.

While it may not be possible to calculate K^n frames to compute the density function of the workspace, it is often feasible to compute $K^{n/2}$ frames for each of the two segments. For the example discussed earlier, this would be on the order of billions of arithmetic calculations, which can be done easily in less than an hour on a not-so-sophisticated computer. The density of the whole workspace is then generated by the convolution of these two functions:

$$(f_1 * f_2)(h) = \int_{SE(3)} f_1(g) f_2\left(g^{-1} \circ h\right) d\mu(g) = f_3(h) .$$

The inverse problem which is of interest is to design the distal end of a manipulator (find $f_2(g)$) for given proximal end ($f_1(g)$ specified), so that the workspace of the combination comes as close as possible to the specified function $f_3(g)$. The most natural way to solve this is to use the Fourier transform of functions on $SE(3)$. In order to do this the matrix elements of irreducible unitary representations must first be generated, as is done in Section 3.

2.2 Statistical Mechanics of Wormlike Polymers

In theoretical and computational polymer science, the probability density function describing the frequency of occurrence of position and/or orientation of one end of a polymer chain relative to the other has received considerable attention (see e.g., [12, 38]). For a stiff polymer such as DNA, a diffusion equation of the form [7]

$$\left(\frac{\partial}{\partial L} - \frac{1}{2}\sum_{k,l=1}^{3} D_{lk}\tilde{X}_l^R \tilde{X}_k^R - \sum_{l=1}^{3} d_l \tilde{X}_l^R + \tilde{X}_6^R\right) f = 0 \tag{2.1}$$

describes the evolution of the probability density function from the proximal end of the polymer where $f(g, 0) = \delta(g)$ to the distal end where the value $f(g, 1)$ is attained (we are assuming the length is normalized).

The differential operators $\tilde{X}_i^R$ are defined as

$$\tilde{X}_i^R f(H) = \left.\frac{df\left(H \circ \exp\left(t\tilde{X}_i\right)\right)}{dt}\right|_{t=0} \tag{2.2}$$

where for the motion group $H = H(g) \in SE(3)$ and

$$\tilde{X}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{X}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{X}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{X}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{X}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{X}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These correspond to infinitesimal rotations and translations about the 1, 2, and 3 axes.

As a concrete example, the Yamakawa helical wormlike model of DNA specifies diffusion constants of the form [38]:

$$D = \begin{pmatrix} \alpha_0^{-1} & 0 & 0 \\ 0 & \alpha_0^{-1} & 0 \\ 0 & 0 & \beta_0^{-1} \end{pmatrix} ; \quad \mathbf{d} = -\begin{pmatrix} 0 \\ \kappa_0 \\ \tau_0 \end{pmatrix} ;$$

where α_0 and β_0 are stiffness parameters and κ_0 and τ_0 specify the prefered helical shape of the chain in the static state.

In polymer science, (2.1) is usually stated without group-theoretic notation, and solving for the pdf $f(g, 1)$ is considered a very difficult problem. For this reason, various techniques for computing moments of $f(g, 1)$ have been devised, though it would be desirable to have $f(g, 1)$ [26, 38].

With an appropriate concept of Fourier transform, the differential operators $\tilde{X}_i^R$ acting on functions on the group $SE(3)$ may be transformed to algebraic operations in Fourier space, and hence in principle (2.1) can be solved using matrix methods. The remainder of this article is devoted to the review of the $SE(3)$-Fourier transform and derivation of associate operational properties that can be used to solve (2.1).

3. Unitary Representations of the Motion Group

In this section we consolidate material presented in [28, 33, 32, 17, 35] to find a complete set of irreducible unitary representations of the motion group.

3.1 Constructing Unitary Representations

We start by constructing representations of the motion group in the space of functions $L^2(\hat{\mathbf{T}}, V)$, where $\hat{\mathbf{T}}$ is the dual space of the $\mathbb{R}^3$ subgroup, and $V \simeq \mathbf{C}$ is the space of complex values of these functions. Functions $\hat{\varphi}(\mathbf{p}) \in L^2(\hat{\mathbf{T}}, V)$, correspond to the Fourier transforms of the functions $\varphi(\mathbf{r}) \in L^2(\mathbf{T}, V)$, where $\mathbf{T} = \mathbb{R}^3$

$$\hat{\varphi}(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbf{T}} e^{-i\mathbf{p}\cdot\mathbf{r}} \varphi(\mathbf{r})\, d\mathbf{r} . \tag{3.1}$$

Henceforth we will drop the hat over φ and use the notation $\varphi(\mathbf{p})$ for the Fourier transform as well as for the function $\varphi(\mathbf{r})$ since it is clear which is being considered from the argument.

The rotation subgroup $SO(3)$ of the motion group acts on $\hat{\mathbf{T}}$ by rotations, so $\hat{\mathbf{T}}$ is divided into orbits S_p, where S_p are S^2 spheres of radius $p = \mid \mathbf{p} \mid$. The translation operator acts on $\varphi(\mathbf{p})$ as

$$(U(\mathbf{a}, I)\varphi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}\, \varphi(\mathbf{p}) . \tag{3.2}$$

Therefore, the irreducible representations of the motion group may be built on spaces $L^2(S_p, V)$, with the inner product defined as

$$(\varphi_1, \varphi_2) = \int_0^{\pi} \int_0^{2\pi} \overline{\varphi_1(\mathbf{p})}\, \varphi_2(\mathbf{p})\ \sin\theta\, d\theta\, d\phi , \tag{3.3}$$

where $\mathbf{p} = (p\ \sin\theta\ \cos\phi,\ p\ \sin\theta\ \sin\phi,\ p\ \cos\theta)$, and $p > 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$.

The inner product (φ_1, φ_2) is invariant with respect to transformations

$$\varphi(\mathbf{p}) \rightarrow e^{i\alpha}\, \varphi\left(A^{-1}\mathbf{p}\right) , \tag{3.4}$$

where $A \in SO(3)$, so the space V is invariant with respect to U(1) rotations (which are isomorphic to $SO(2)$ rotations for $0 \leq \alpha \leq 2\pi$).

The parameter α in (3.4) may, in general, depend on p and group element $A \in SO(3)$. In this case, different functions $\alpha_s(p, A)$ (where s enumerates the irreducible representations of $SO(2)$), which are nonlinear functions of group element A, correspond to different irreducible representations of the motion group. Thus functions $\varphi(\mathbf{p})$ may have different *internal* properties with respect to rotations.

With the help of functions $\alpha_s(p, A)$ we may construct the representations of $G = SE(3) \simeq \hat{\mathbf{T}} \rhd SO(3)$ from representations of its subgroup $G' = \hat{\mathbf{T}} \rhd SO(2)$. These representations are called *induced* representations of G, denoted by $G' \uparrow G$.

A formal definition of the induced representations (see, for example, [8, 25, 17]) $\Delta(H) \uparrow G$, where $\Delta(H)$ are representations of subgroup H of group G, is the following.

Definition. Let V be the space of complex values of functions $\varphi(\sigma)$, where $\sigma \in G/H$. The action of operators $U(g) \in \Delta \uparrow G$ ($g \in G$) (where Δ is a representation of subgroup H of group G, and Δ acts in V) is such that

$$(U(g)\varphi)(\sigma) = \Delta\left(s_\sigma^{-1}\, g\, s_{g^{-1}\sigma}\right) \varphi\left(g^{-1}\sigma\right) , \tag{3.5}$$

where s_σ is an arbitrary representative of the coset $\sigma \in G/H$.

In our case (we disregard for the moment the translation group $\hat{\mathbf{T}}$), $G = SO(3)$, $H = SO(2)$, and $\sigma = \mathbf{p} \in S_p \simeq SO(3)/SO(2)$.

To construct the representations of the motion group explicitly we choose a particular vector $\hat{\mathbf{p}} = (0, 0, p)$ on each orbit S_p. The vector $\hat{\mathbf{p}}$ is invariant with respect to rotations from the $SO(2)$ subgroup of $SO(3)$

$$\Lambda\,\hat{\mathbf{p}} = \hat{\mathbf{p}}; \qquad \Lambda \in H_{\hat{\mathbf{p}}} = SO(2) , \tag{3.6}$$

where $H_{\hat{\mathbf{p}}}$ is the little group of $\hat{\mathbf{p}}$. For each $\mathbf{p} \in S_p$ we may find $R_\mathbf{p} \in SO(3)/SO(2)$, such that

$$R_\mathbf{p}\,\hat{\mathbf{p}} = \mathbf{p} .$$

Then for any $A \in SO(3)$ one may check that

$$\left(R_\mathbf{p}^{-1}\, A\, R_{A^{-1}\mathbf{p}}\right) \hat{\mathbf{p}} = \hat{\mathbf{p}} .$$

Therefore, $Q(\mathbf{p}, A) = (R_\mathbf{p}^{-1}\, A\, R_{A^{-1}\mathbf{p}}) \in H_{\hat{\mathbf{p}}}$. The representations of $H_{\hat{\mathbf{p}}}$ may be taken to be of the form

$$\Delta_s : \ \phi \to e^{is\phi} ; \ 0 \leq \phi \leq 2\pi ;$$

and $s = 0, \pm 1, \pm 2, \ldots$.

Thus, we may construct the induced representation $(\hat{\mathbf{T}} \rhd \Delta_s(H_{\hat{\mathbf{p}}})) \uparrow SE(3)$ of the motion group from the representations of its subgroup $\hat{\mathbf{T}} \rhd H_{\hat{\mathbf{p}}}$.

Definition. The unitary representations $U^s(\mathbf{a}, A)$ of $SE(3)$, which act on the space of functions $\varphi(\mathbf{p})$ with the inner product (3.3), are defined by

$$\left(U^s(\mathbf{a}, A)\varphi\right)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}\, \Delta_s\left(R_\mathbf{p}^{-1}\, A\, R_{A^{-1}\mathbf{p}}\right) \varphi\left(A^{-1}\mathbf{p}\right) , \tag{3.7}$$

where $A \in SO(3)$, Δ_s are representations of $H_{\hat{\mathbf{p}}}$ and $s = 0, \pm 1, \pm 2, \ldots$

Each representation, characterized by $p = \mid \mathbf{p} \mid$ and s, is irreducible (they, however, become reducible if we restrict $SE(3)$ to $SO(3)$, i.e., when $a = 0$). They are unitary, because $(U^s(\mathbf{a}, A)\varphi_1, U^s(\mathbf{a}, A)\varphi_2) = (\varphi_1, \varphi_2)$.

Representations (3.7), which we denote below by $U^s(g,p)$, satisfy the homomorphism property

$$U^s(g_1 \circ g_2, p) = U^s(g_1, p) \cdot U^s(g_2, p) ,$$

where $\circ$ is the group operation. The corresponding multiplication law for the $Q(\mathbf{p}, A)$ factors is [37]

$$Q(\mathbf{p}, A)\, Q\left(A^{-1}\mathbf{p}, A^{-1}B\right) = Q(\mathbf{p}, B) . \tag{3.8}$$

3.2 Matrix Elements

To obtain the matrix elements of these unitary representations we use the group property

$$U^s(\mathbf{a}, A) = U^s(\mathbf{a}, I) \cdot U^s(0, A) . \tag{3.9}$$

The basis eigenfunctions of the irreducible representations (3.7) of $SE(3)$ may be enumerated by the integer numbers l, m (for each s and p). We note that the values $l(l+1), m, ps$ and $-p^2$ correspond to the eigenvalues of the generators $\mathbf{J}^2, J^3, \mathbf{P}\cdot\mathbf{J}, \mathbf{P}\cdot\mathbf{P}$ (where $J^i, P^i, i = 1, 2, 3$ are generators of rotation and translation) of the Lie algebra $SE(3)$ (see [14, 28]) of the motion group $SE(3)$, which may be diagonalized simultaneously (i.e., they commute). The restrictions for the l, m, s numbers are $l \geq \mid s \mid$; $l \geq \mid m \mid$.

The basis functions may be expressed in the form [28]

$$h^l_{m\,s}(\theta, \phi) = Q^l_{s,m}(\cos\theta)\, e^{i(m+s)\phi} \tag{3.10}$$

where

$$Q^l_{-s,m}(\cos\theta) = (-1)^{l-s}\sqrt{\frac{2l+1}{4\pi}}\, P^l_{s\,m}(\cos\theta) ,$$

and generalized Legendre polynomials $P^l_{m\,s}(\cos\theta)$ are given as in Vilenkin [35].

It may be shown that these basis functions are transformed under the rotations $h^l_{m\,s}(\mathbf{p}) \rightarrow \Delta_s(Q(\mathbf{p}, A))\, h^l_{m\,s}(A^{-1}\mathbf{p})$ as in [7]

$$\left(U^s(0, A)\, h^l_{m\,s}\right)(\mathbf{p}) = \sum_{n=-l}^{l} U^l_{n\,m}(A) h^l_{n\,s}(\mathbf{p}) , \tag{3.11}$$

where the matrix elements $U^l_{n\,m}(A)$ are

$$U^l_{m\,n}(A) = e^{-im\alpha}\,(-1)^{n-m}\, P^l_{m\,n}(\cos\beta)\, e^{-in\gamma} , \tag{3.12}$$

where α, β, γ are $z - x - z$ Euler angles of the rotation. We note that the rotation matrix elements do not depend on s.

The translation matrix elements are given by the integral [28]

$$\begin{aligned}\left(h^{l'}_{m'\,s}, U^s(\mathbf{a}, I) h^l_{m\,s}\right) &= \left[l', m' \mid p, s \mid l, m\right](\mathbf{a}) \\ &= \int_0^{\pi}\int_0^{2\pi} Q^{l'}_{s,m'}(\cos\theta) e^{-i(m'+s)\phi}\, e^{-i\mathbf{p}\cdot\mathbf{a}}\, Q^l_{s,m}(\cos\theta) e^{i(m+s)\phi}\, \sin\theta\, d\theta\, d\phi . \end{aligned} \tag{3.13}$$

These are written in closed form as [28]

$$\begin{aligned}&\left[l', m' \mid p, s \mid l, m\right](\mathbf{a}) \\ &= (4\pi)^{1/2} \sum_{k=|l'-l|}^{l'+l} i^k \sqrt{\frac{(2l'+1)(2k+1)}{(2l+1)}}\, j_k(p\,a)\, C\left(k, 0; l', s \mid l, s\right) \\ &\cdot C\left(k, m-m'; l', m' \mid l, m\right)\, Y_k^{m-m'}(\theta, \phi) , \end{aligned} \tag{3.14}$$

where in (3.14) θ, ϕ are the polar and azimuthal angles of $\mathbf{a}$, $j_k(x)$ are half-integer Bessel functions, $Y_k^m(\theta, \phi)$ are spherical harmonics, and $C(k, m-m'; l', m' \mid l, m)$ are Clebsch-Gordan coefficients (see, for example, [23]).

Finally, using the group property (3.9), the matrix elements of the unitary representation $U^s(g, p)$ (3.7) (for $s = 0, \pm 1, \pm 2, \ldots$) are expressed as

$$U^s_{l',m';l,m}(\mathbf{a}, A; p) = \sum_{j=-l}^{l} \left[l', m' \mid p, s \mid l, j\right](\mathbf{a})\, U^l_{j\,m}(A)\,. \tag{3.15}$$

We note that

$$\overline{U^s_{l',m';l,m}(\mathbf{a}, A; p)} = (-1)^{(l'-l)}(-1)^{(m'-m)} U^s_{l',-m';l,-m}(\mathbf{a}, A; p)\,. \tag{3.16}$$

Because (3.14) contains only half-integer Bessel functions, all matrix elements may be expressed in terms of elementary functions.

4. Fourier Transform

Here we define the Fourier transform of functions $f(\mathbf{a}, A) \in L^2(SE(3))$. The inner product of functions is given by

$$(f, g) = \int_{SE(3)} \overline{f(\mathbf{a}, A)}\, g(\mathbf{a}, A)\, dA\, d^3\mathbf{a}\,. \tag{4.1}$$

To define the Fourier transform for functions on $SE(3)$ we have to use a complete orthogonal basis for functions on this group. The completeness of matrix elements (3.15) is based on the completeness of the rotation matrix elements $U^l_{mn}(A)$ on $SO(3)$ [28]. Using the unitary representations $U(g, p)$ (3.7) (for $s = 0, \pm 1, \pm 2 \ldots$), we may define the Fourier transform of functions on the motion group.

Definition. For any integrable complex-valued function $f(\mathbf{a}, A)$ on $SE(3)$ we define the Fourier transform as

$$\mathcal{F}(f) = \hat{f}(p) = \int_{SE(3)} f(g) U\left(g^{-1}, p\right) d\mu(g)$$

where $g \in SE(3)$ and $d\mu(g) = dA\, d^3\mathbf{a}$ where $dA = \frac{1}{8\pi^2} \sin\beta d\alpha d\beta d\gamma$ is the normalized bi-invariant measure for $SO(3)$ and $d^3\mathbf{a} = da_1 da_2 da_3$.

The matrix elements of the transform are given in terms of matrix elements (3.15) as

$$\hat{f}^s_{l',m';l,m}(p) = \int_{SE(3)} f(\mathbf{a}, A)\, \overline{U^s_{l,m;l',m'}(\mathbf{a}, A; p)}\, dA\, d^3\mathbf{a} \tag{4.2}$$

where we have used the unitarity property.

The inverse Fourier transform is defined by

$$f(g) = \mathcal{F}^{-1}\left(\hat{f}\right) = \frac{1}{2\pi^2} \int_{SE(3)} \mathrm{Tr}\left(\hat{f}(p) U(g, p)\right) p^2\, dp\,. \tag{4.3}$$

Explicitly

$$f(\mathbf{a}, A) = \frac{1}{2\pi^2} \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^{l} \int_0^{\infty} p^2\, dp\, \hat{f}^s_{l,m;l',m'}(p) U^s_{l',m';l,m}(\mathbf{a}, A; p)\,. \tag{4.4}$$

Convolution of functions. Recall that the convolution integral of functions $f_1, f_2 \in L^2(SE(3))$ may be defined as

$$(f_1 * f_2)(g) = \int_{SE(3)} f_1(h)\, f_2\left(h^{-1} \circ g\right) d\mu(h)\,. \tag{4.5}$$

One of the most powerful properties of the Fourier transform of functions on $\mathbb{R}^N$ is that the Fourier transform of the convolution of two functions is the product of the Fourier transform of the functions. This property persists also for the convolution of functions on the group. Namely,

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_2)\,\mathcal{F}(f_1) \tag{4.6}$$

or, in matrix form

$$(\mathcal{F}(f_1 * f_2))^s_{l',m';l,m}(p) = \sum_{j=|s|}^{\infty} \sum_{k=-j}^{j} \left(\hat{f}_2\right)^s_{l',m';j,k}(p) \left(\hat{f}_1\right)^s_{j,k;l,m}(p)\,.$$

Plancherel equality. This form of the Plancherel equality is valid

$$\begin{aligned}
&\int_{SE(3)} \mid f(\mathbf{a}, A) \mid^2 \, dA d^3\mathbf{a} \\
&= \frac{1}{2\pi^2} \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^{l} \int_0^{\infty} \left|\hat{f}^s_{l',m';l,m}(p)\right|^2 p^2\, dp \\
&= \frac{1}{2\pi^2} \int_0^{\infty} \left\|\hat{f}(p)\right\|_2^2 p^2\, dp\,,
\end{aligned} \tag{4.7}$$

where the Hilbert-Schmidt norm of $\hat{f}(p)$ is given by

$$\left\|\hat{f}(p)\right\|_2^2 = \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^{l} \left|\hat{f}^s_{l',m';l,m}(p)\right|^2 .$$

Symmetries. For the real function $f(\mathbf{a}, A)$ we note a symmetry property of the Fourier transform, which follows from symmetry (3.16) of the matrix elements

$$\overline{\hat{f}^s_{l',m';l,m}(p)} = (-1)^{(l'-l)} (-1)^{(m'-m)} \hat{f}^s_{l',-m';l,-m}(p)\,. \tag{4.8}$$

Contraction of indices. It is convenient to rewrite the 4-index Fourier transform matrix element $\hat{f}^s_{l',m';l,m}(p)$ as a 2-index matrix $\hat{f}^s_{ij}(p)$. To satisfy the matrix product definition we define:

$$\hat{f}^s_{l',m';l,m}(p) = \hat{f}^s_{ij}(p)\,, \tag{4.9}$$

where $i = l'(l'+1) + m' - s^2 + 1$; $j = l(l+1) + m - s^2 + 1$.

5. Operational Properties

Properties of translation differential operators. It may be observed from the integral representations (3.13) and (3.15) that the matrix elements of the motion group satisfy the relation

$$\nabla^2_{\mathbf{a}} U^s_{l',m';l,m}(\mathbf{a}, A; p) = \left(-p^2\right) U^s_{l',m';l,m}(\mathbf{a}, A; p) \tag{5.1}$$

where $\nabla^2_{\mathbf{a}}$ is the Laplacian with respect to $\mathbf{a}$. This equation leads to the Fourier transform property

$$\mathcal{F}\left(\nabla^2_{\mathbf{a}} f(\mathbf{a}, A)\right) = \left(-p^2\right) \hat{f}(p) . \tag{5.2}$$

For the function $f(\mathbf{a}, A)$, rapidly decreasing as $a = \mid \mathbf{a} \mid \to \infty$ and less singular than $\frac{1}{a}$ as $a \to 0$, the following relation is valid

$$\mathcal{F}\left(a \frac{\partial}{\partial a} f(\mathbf{a}, A)\right) = -\left(p \frac{d\hat{f}(p)}{dp} + 3\,\hat{f}(p)\right) . \tag{5.3}$$

This may be proven using the equation

$$\frac{\partial}{\partial a} f(\mathbf{a}, A) = \frac{1}{2\pi^2} \int_0^\infty \operatorname{Tr}\left(\hat{f}(p) \frac{\partial}{\partial a} U(g, p)\right) p^2\, dp , \tag{5.4}$$

the fact that matrix elements of $U(g, p)$ are functions of product (pa) [it may be seen from (3.14)], and integrating (5.4) by parts.

The more general property

$$\mathcal{F}\left(\frac{1}{a^{n-1}} \frac{\partial}{\partial a}\left(a^n f(\mathbf{a}, A)\right)\right) = (n-3)\,\hat{f}(p) - p \frac{d\hat{f}(p)}{dp}$$

follows from (5.3).

We mention also the related property

$$\int_{SE(3)} \left| a \frac{\partial}{\partial a} f(\mathbf{a}, A)\right|^2 dA\, d^3\mathbf{a} = \frac{1}{2\pi^2} \int_0^\infty \left\| p \frac{d\hat{f}(p)}{dp}\right\|_2^2 p^2\, dp ,$$

which follows from the Plancherel equality (4.7) and (5.3).

The equality $i \frac{\partial}{\partial a_3} e^{-i\mathbf{p}\cdot\mathbf{a}} = p \cos\theta\, e^{-i\mathbf{p}\cdot\mathbf{a}} = -p\sqrt{\frac{4\pi}{3}}\, Q^1_{00}(\cos\theta)\, e^{-i\mathbf{p}\cdot\mathbf{a}}$ (where θ is the polar angle of $\mathbf{p}$) and the explicit expressions for the Clebsch-Gordan coefficients lead to the relation

$$\begin{aligned} & i \frac{\partial}{\partial a_3} U^s_{l',m';l,m}(\mathbf{a}, A; p) \\ & = -p \left(\frac{\left(l'^2 - m'^2\right)\left(l'^2 - s'^2\right)}{(2l'+1)(2l'-1)\,l'^2}\right)^{1/2} U^s_{l'-1,m';l,m}(\mathbf{a}, A; p) \\ & \quad - p \frac{m' s}{l'(l'+1)} U^s_{l',m';l,m}(\mathbf{a}, A; p) \\ & \quad - p \left(\frac{\left((l'+1)^2 - m'^2\right)\left((l'+1)^2 - s'^2\right)}{(2l'+1)(l'+1)^2(2l'+3)}\right)^{1/2} U^s_{l'+1,m';l,m}(\mathbf{a}, A; p) . \end{aligned} \tag{5.5}$$

We may get from this relation

$$\begin{aligned} \mathcal{F}\left(i \frac{\partial}{\partial a_3} f(\mathbf{a}, A)\right) = & -p \left(\frac{\left((l+1)^2 - m^2\right)\left((l+1)^2 - s^2\right)}{(2l+1)(l+1)^2(2l+3)}\right)^{1/2} \hat{f}^s_{l',m';l+1,m}(p) \\ & - p \frac{m s}{l(l+1)} \hat{f}^s_{l',m';l,m}(p) \\ & - p \left(\frac{\left(l^2 - m^2\right)\left(l^2 - s^2\right)}{(2l+1)(2l-1)l^2}\right)^{1/2} \hat{f}^s_{l',m';l-1,m}(p) . \end{aligned}$$

For operators $P^{\pm} = i\frac{\partial}{\partial a_1} \pm \frac{\partial}{\partial a_2}$, which act on the exponent as $P^{\pm}e^{-i\mathbf{p}\cdot\mathbf{a}} = \pm\sqrt{\frac{8\pi}{3}}$ $Q^1_{0,\pm 1}(\cos\theta)\, e^{\mp i\phi}$ (θ, ϕ are polar and azimuthal angles of $\mathbf{p}$), we have the relations

$$\begin{aligned}
&P^+ U^s_{l',m';l,m}(\mathbf{a}, A; p) \\
&\quad = -p\left(\frac{(l'-1-m')(l'-m')(l'^2-s'^2)}{(2l'+1)(2l'-1)l'^2}\right)^{1/2} U^s_{l'-1,m'+1;l,m}(\mathbf{a}, A; p) \\
&\qquad - p\,\frac{\sqrt{(l'-m')(l'+1+m')}\,s}{l'(l'+1)}\, U^s_{l',m'+1;l,m}(\mathbf{a}, A; p) \\
&\qquad + p\left(\frac{(l'+1+m')(l'+2+m')\left((l'+1)^2-s'^2\right)}{(2l'+1)(l'+1)^2(2l'+3)}\right)^{1/2} U^s_{l'+1,m'+1;l,m}(\mathbf{a}, A; p)
\end{aligned}$$

and

$$\begin{aligned}
&P^- U^s_{l',m';l,m}(\mathbf{a}, A; p) \\
&\quad = -p\left(\frac{(l'-1+m')(l'+m')(l'^2-s'^2)}{(2l'+1)(2l'-1)l'^2}\right)^{1/2} U^s_{l'-1,m'-1;l,m}(\mathbf{a}, A; p) \\
&\qquad - p\,\frac{\sqrt{(l'+m')(l'+1-m')}\,s}{l'(l'+1)}\, U^s_{l',m'-1;l,m}(\mathbf{a}, A; p) \\
&\qquad - p\left(\frac{(l'+1-m')(l'+2-m')\left((l'+1)^2-s'^2\right)}{(2l'+1)(l'+1)^2(2l'+3)}\right)^{1/2} U^s_{l'+1,m'-1;l,m}(\mathbf{a}, A; p)\,.
\end{aligned}$$

These relations yield the operational properties

$$\begin{aligned}
&\mathcal{F}\left(P^+ f(\mathbf{a}, A)\right) \\
&\quad = -p\left(\frac{(l+1-m)(l+2-m)\left((l+1)^2-s^2\right)}{(2l+1)(l+1)^2(2l+3)}\right)^{1/2} \hat{f}^s_{l',m';l+1,m-1}(p) \\
&\qquad - p\,\frac{\sqrt{(l+m)(l+1-m)}\,s}{l(l+1)}\, \hat{f}^s_{l',m';l,m-1}(p) \\
&\qquad + p\left(\frac{(l-1+m)(l+m)(l^2-s^2)}{(2l+1)(2l-1)l^2}\right)^{1/2} \hat{f}^s_{l',m';l-1,m-1}(p)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{F}(P^- f(\mathbf{a}, A)) \\
&\quad = p\left(\frac{(l+1+m)(l+2+m)((l+1)^2-s^2)}{(2l+1)(l+1)^2(2l+3)}\right)^{1/2} \hat{f}^s_{l',m';l+1,m+1}(p) \\
&\qquad - p\,\frac{\sqrt{(l-m)(l+1+m)}\,s}{l(l+1)}\, \hat{f}^s_{l',m';l,m+1}(p) \\
&\qquad - p\left(\frac{(l-1-m)(l-m)(l^2-s^2)}{(2l+1)(2l-1)l^2}\right)^{1/2} \hat{f}^s_{l',m';l-1,m+1}(p)\,.
\end{aligned}$$

We note that $\hat{f}^s_{l',m';l,m}(p) = 0$ if $|\, m'(m) \,|> l'(l)$.

Using integration by parts together with the property (5.2) it may be shown that

$$\begin{aligned}\int_{SE(3)} |\nabla_{\mathbf{a}} f(g)|^2 \, d\mu(g) &= \int_{SE(3)} \left(f(g), -\nabla_{\mathbf{a}}^2 f(g)\right) d\mu(g) \\ &= \frac{1}{2\pi^2} \int_0^\infty \operatorname{Tr}\left(p^2 \hat{f}^\dagger(p) \hat{f}(p)\right) p^2 \, dp \,. \end{aligned} \tag{5.6}$$

Properties of rotation differential operators. From the fact that [35, 33]

$$\nabla_A^2 \, U_{mn}^l(A) = (-l(l+1)) \, U_{mn}^l(A) \,, \tag{5.7}$$

where the Laplacian operator on $SO(3)$ is given by

$$\nabla_A^2 = \frac{1}{\sin\beta} \frac{\partial}{\partial\beta}\left(\sin\beta \frac{\partial}{\partial\beta}\right) + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta \frac{\partial^2}{\partial\alpha\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right) \tag{5.8}$$

(recall that α, β, γ are $z-x-z$ Euler angles of A), it follows that

$$\mathcal{F}\left(\nabla_A^2 f(\mathbf{a}, A)\right)^s_{l',m';l,m} = \left(-l'\left(l'+1\right)\right) \hat{f}^s_{l',m';l,m}(p) \,. \tag{5.9}$$

The straightforward relation

$$\mathcal{F}\left(i\frac{\partial}{\partial\gamma} f(\mathbf{a}, A)\right)^s_{l',m';l,m} = m' \, \hat{f}^s_{l',m';l,m}(p)$$

follows from (3.12) and (3.15) and Fourier transform definition (4.3).

The equations [14]

$$\begin{aligned} H_- U_{mn}^l(A) &= i\sqrt{(l+n)(l-n+1)} \, U_{m,n-1}^l(A) \\ H_+ U_{mn}^l(A) &= -i\sqrt{(l-n)(l+n+1)} \, U_{m,n+1}^l(A) \,, \end{aligned}$$

where

$$\begin{aligned} H_- &= e^{i\gamma}\left(-\cot\beta \frac{\partial}{\partial\gamma} + \frac{1}{\sin\beta}\frac{\partial}{\partial\alpha} + i\frac{\partial}{\partial\beta}\right) ; \\ H_+ &= e^{-i\gamma}\left(\cot\beta \frac{\partial}{\partial\gamma} - \frac{1}{\sin\beta}\frac{\partial}{\partial\alpha} + i\frac{\partial}{\partial\beta}\right) ; \end{aligned}$$

lead to the relations

$$\begin{aligned} \mathcal{F}\left(H_- f(\mathbf{a}, A)\right)^s_{l',m';l,m} &= i\sqrt{\left(l'+m'+1\right)\left(l'-m'\right)} \, \hat{f}^s_{l',m'+1;l,m}(p) \,, \\ \mathcal{F}\left(H_+ f(\mathbf{a}, A)\right)^s_{l',m';l,m} &= -i\sqrt{\left(l'-m'+1\right)\left(l'+m'\right)} \, \hat{f}^s_{l',m'-1;l,m}(p) \,. \end{aligned}$$

6. Analytical Examples

Here we give some examples which illustrate the Fourier transform and inverse Fourier transform and allow us to perform analytical calculations.

First, let us consider rapidly decreasing spherically symmetric functions

$$f(\mathbf{r}, A) = f(r) \,.$$

Here we have used $\mathbf{r}$ to denote translation rather than $\mathbf{a}$. Because this function does not depend on the Euler angles of rotation, only $U^s_{l',m';0,0}(\mathbf{r}, A)$ elements contribute to the Fourier transform. Moreover, $s = 0$ because $|\, s \,| \leq l, l'$. Finally, examining the expression (3.14) and using the fact that

$$\int_0^\pi \int_0^{2\pi} Y_l^m \sin\theta \, d\theta \, d\phi = \sqrt{4\pi}\, \delta_{l0}\, \delta_{m0} \,,$$

we see that only the element $U^0_{0,0;0,0}(\mathbf{r}, A; p)$ contributes to the Fourier transform.

For the function $f(r) = e^{-r}$ the Fourier transform gives

$$\begin{aligned} \hat{f}^0_{0,0;0,0}(p) &= \int_{SE(3)} f(r) \overline{U^0_{0,0;0,0}(\mathbf{r}, A; p)}\, d^3\mathbf{r}\, dA \\ &= 4\pi \int_0^\infty e^{-r} \frac{\sin(pr)}{p} r\, dr = \frac{8\pi}{(1+p^2)^2} \,, \end{aligned} \tag{6.1}$$

where we have used the fact that $\int_{SO(3)} dA = 1$.

The inverse Fourier transform reproduces the original function

$$\begin{aligned} f(r) &= \frac{1}{2\pi^2} \int_0^\infty \hat{f}^0_{0,0;0,0}(p) U^0_{0,0;0,0}(\mathbf{r}, A; p)\, p^2 dp \\ &= \frac{1}{2\pi^2} \int \frac{8\pi}{\left(1+p^2\right)^2} \frac{\sin(pr)}{r}\, p\, dp = e^{-r} \,. \end{aligned}$$

For the function $f(r) = e^{-r^2}$ we have for the Fourier transform $\hat{f}(p) = (\pi)^{3/2} e^{-p^2/4}$. The inverse Fourier transform gives the original function.

Another example is the function

$$f(\mathbf{r}, A) = e^{-r} \cos\theta \cos\beta$$

where θ is the polar angle of $\mathbf{r}$ and β is Euler angle (around the x-axis) of rotation A. Because $U^1_{00}(A) = \cos\beta$, it may be shown from (3.14) and (3.15) that only $\hat{f}^0_{1,0;0,0}(p)$; $\hat{f}^0_{1,0;1,0}(p)$; $\hat{f}^0_{1,0;2,0}(p)$; $\hat{f}^1_{1,0;1,0}(p)$; $\hat{f}^1_{1,0;2,0}(p)$; $\hat{f}^{-1}_{1,0;1,0}(p)$; $\hat{f}^{-1}_{1,0;2,0}(p)$ can give nonzero contributions. Direct computations show that the Fourier transform elements are (we show only nonzero matrix elements)

$$\begin{aligned} \hat{f}^0_{1,0;0,0}(p) &= -\frac{8i\pi}{3\sqrt{3}} \frac{p}{\left(1+p^2\right)^2} \,; \\ \hat{f}^0_{1,0;2,0}(p) &= -\frac{16i\pi}{\sqrt{135}} \frac{p}{\left(1+p^2\right)^2} \,; \\ \hat{f}^1_{1,0;2,0}(p) &= -\frac{8i\pi}{3\sqrt{5}} \frac{p}{\left(1+p^2\right)^2} \,; \\ \hat{f}^{-1}_{1,0;2,0}(p) &= -\frac{8i\pi}{3\sqrt{5}} \frac{p}{\left(1+p^2\right)^2} \,. \end{aligned} \tag{6.2}$$

For the inverse Fourier transform we obtain the following expression for the trace in equation (4.3)

$$\operatorname{Tr}\left(\hat{f}(p) U(g, p)\right) = 8\pi \cos\theta \cos\beta \frac{(\sin(pr) - pr\cos(pr))}{\left(1+p^2\right)^2 pr^2} \,.$$

The p integration in (4.3) reproduces the original function.

We computed also the Fourier transform of the function

$$f(\mathbf{r}, A) = r^2 e^{-r} \cos\theta \, \cos\beta \, .$$

The nonzero matrix elements are

$$\begin{aligned}
\hat{f}^{0}_{1,0;0,0}(p) &= \frac{32 i\pi}{3\sqrt{3}} \, \frac{p\,(p^2-5)}{\left(1+p^2\right)^4} \, ; \\
\hat{f}^{0}_{1,0;2,0}(p) &= \frac{64 i\pi}{\sqrt{135}} \, \frac{p\,(p^2-5)}{\left(1+p^2\right)^4} \, ; \\
\hat{f}^{1}_{1,0;2,0}(p) &= \frac{32 i\pi}{3\sqrt{5}} \, \frac{p\,(p^2-5)}{\left(1+p^2\right)^4} \, ; \\
\hat{f}^{-1}_{1,0;2,0}(p) &= \frac{32 i\pi}{3\sqrt{5}} \, \frac{p\,(p^2-5)}{\left(1+p^2\right)^4} \, .
\end{aligned} \tag{6.3}$$

The inverse Fourier transform gives the original function.

We chop all zero elements for $l(l') > 2$. Therefore, after contraction of four indices to two using (4.9), the Fourier transform of the examples (6.3) and (6.2) may be written as a block-diagonal matrix

$$\hat{F} = \begin{bmatrix} \boxed{\hat{F}_{-1}} & & \\ & \boxed{\hat{F}_0} & \\ & & \boxed{\hat{F}_1} \end{bmatrix} . \tag{6.4}$$

The nonzero blocks are the 9×9 matrix $\hat{F}_0$ and two 8×8 matrices $\hat{F}_{-1}$, $\hat{F}_1$ (lower indices correspond to s index). Using (4.9) these matrices may be depicted as

$$\hat{F}_0 = \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & & \vdots \\ \hat{f}^{0}_{31} & \dots & \hat{f}^{0}_{37} & 0 \\ \vdots & & & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \tag{6.5}$$

where $\hat{f}^{0}_{31} = \hat{f}^{0}_{1,0;0,0}(p)$, $\hat{f}^{0}_{37} = \hat{f}^{0}_{1,0;2,0}(p)$. The other matrices are

$$\hat{F}_{\pm 1} = \begin{bmatrix} 0 & \dots & 0 & 0 \\ 0 & \dots & \hat{f}^{\pm 1}_{26} & 0 \\ \vdots & & & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} \tag{6.6}$$

where $\hat{f}^{\pm 1}_{26} = \hat{f}^{\pm 1}_{1,0;2,0}(p)$.

7. Application of the Fourier Transform to the Solution of Convolution Equations

In this section we apply Fourier analysis on $SE(3)$ to the solution of the convolution equation (1.1).

7.1 Regularization of the Convolution Equation

Using the property (4.6), the convolution equation may be written in the form

$$\hat{f}_2 \hat{f}_1 = \hat{f}_3 \,, \tag{7.1}$$

where $\hat{f}_i$ denotes the $SE(3)$-Fourier transform of function f_i.

In general, the Fourier transform can contain an infinite number of harmonics l, l' and block elements s. We assume that the contribution of the higher (rapidly oscillating) harmonics can be neglected, and truncate the Fourier transform at some $l = l'$ for each block and take all nonzero blocks for $\mid s \mid \leq l$. Thus, the problem may be reduced to the solution of the matrix equation (7.1). If the functions are "band limited" (i.e., only a finite number of harmonics give contributions to the Fourier transform), the Fourier transform matrices are finite (as in examples in the previous section).

For nonsingular matrix $\hat{f}_1$ the inverse Fourier transform is used to generate the solution:

$$f_2(g) = \mathcal{F}^{-1}\left(\hat{f}_3(p)\hat{f}_1^{-1}(p)\right) . \tag{7.2}$$

However, in practice, $\hat{f}_1(p)$ is usually singular for many or all values of p, and so a means of regularization is required.

This is a perfect application of the operational properties derived in Section 5. One can extend the Tikhonov regularization technique [16], used for integral equations of real-valued argument, to the case of $SE(3)$. That is, instead of solving the original problem, we seek an approximate solution which minimizes the cost function (this is a particular example of first order Tikhonov regularization):

$$\begin{aligned} C \;=\; & \int_{SE(3)} \left(|(f_1 * f_2)(g) - f_3(g)|^2 + \epsilon\, |f_2(g)|^2 + \nu\, |\nabla_{\mathbf{a}} f_2(g)|^2 \right. \\ & \left. + \,\eta \left(f_2(g), -\nabla_A^2 f_2(g) \right) \right) d\mu(g) \end{aligned} \tag{7.3}$$

for small parameters ϵ, ν and η (higher order derivatives may be added for higher order regularization). Here $g = (\mathbf{a}, A)$.

Using the operational properties (5.6) and (5.9) together with the Plancherel equality (4.7), one can convert this cost function into an algebraic expression in the dual (Fourier) space, do algebraic manipulations, and convert back using the inverse transform.

The Fourier transform converts (7.3) into

$$\begin{aligned} C \;=\; & \frac{1}{2\pi^2} \int_0^\infty c\left(\hat{f}_2^\dagger(p), \hat{f}_2(p)\right) p^2\, dp = \frac{1}{2\pi^2} \int_0^\infty \left(\left\| \hat{f}_2(p)\, \hat{f}_1(p) - \hat{f}_3(p) \right\|_2^2 \right. \\ & \left. + \,\epsilon \left\| \hat{f}_2(p) \right\|_2^2 + \nu \left\| p\, \hat{f}_2(p) \right\|_2^2 + \eta\, \mathrm{Tr}\left(\hat{f}_2^\dagger(p)\, \mathcal{A} \hat{f}_2(p) \right) \right) p^2\, dp \,, \end{aligned} \tag{7.4}$$

where $\mathcal{A}_{l',m';l,m} \triangleq l'(l'+1)\delta_{l'l}\,\delta_{m'm}$.

The equation for $\hat{f}_2(p)$, which minimizes the functional C may be found by differentiating $c(\hat{f}_2^\dagger(p), \hat{f}_2(p))$ with respect to $\hat{f}_2^\dagger(p)$ (or $\hat{f}_2(p)$)

$$\frac{\partial c}{\partial \hat{f}_2^\dagger} = 0 \,,$$

(differentiation with respect to $\hat{f}_2(p)$ gives a Hermitian conjugate equation). This equation is written explicitly as

$$\hat{f}_2\left(\hat{f}_1\hat{f}_1^{\dagger} + \left(\epsilon + \nu p^2\right)\mathbf{1}\right) + \eta\mathcal{A}\hat{f}_2 = \hat{f}_3\hat{f}_1^{\dagger}, \tag{7.5}$$

where $\mathbf{1}$ is an appropriately dimensioned identity matrix. After truncating the Fourier transforms and contraction of indices according to (4.9) this equation is analogous to the matrix equation

$$\hat{f}_2\mathcal{B} + \mathcal{A}\hat{f}_2 = \mathcal{D} \tag{7.6}$$

for given matrices $\mathcal{A}, \mathcal{B}, \mathcal{D}$. Methods for solving this equation can be found in [13, 2, 15].

We have to solve equation (7.5) for smaller and smaller values of the parameters ϵ, ν, η. When the solution starts to exhibit unpleasant behavior (the solution shows singular-like growth in some regions and starts to oscillate) the calculations must be stopped. "Physical" arguments may be used in the choice of the particular values of the parameters, i.e., we may want to pay more attention to the magnitude of derivatives of the solution (in which case we increase parameter ν) or we may be interested in the absolute values of the solution (in which case we increase parameter ϵ). The solution for these values of the parameters ϵ, ν, η is an approximation to the solution of the convolution equation (1.1).

7.2 An Example Solution of the Regularized Problem

Here we find a solution which minimizes functional (7.3) for $\eta = 0$ and some values of ϵ and ν. We solve this problem for particular functions $f_1(\mathbf{r}, A) = e^{-r}$ and $f_3(\mathbf{r}, A) = e^{-r}\cos\theta\cos\beta$, where $r = |\,\mathbf{r}\,|$, θ is the polar angle of $\mathbf{r}$, and β is an Euler angle (around the x-axis) of rotation A.

The Fourier transforms of the functions f_1 and f_3 above are given in (6.1) and (6.2). After truncation of zero elements and contraction of indices the Fourier transform of $f_3(g)$ is written in matrix form in (6.4), (6.5), and (6.6). Only the $s = 0$ block of the Fourier transform of $f_1(g)$ is nonzero, and it may be depicted as a 9×9 matrix $\hat{f}_1$ with only one nonzero element $(\hat{f}_1)_{11}(p)$. It is clear that $\hat{f}_1$ does not have an inverse. So, instead of solving (7.1), we look for a solution $\hat{f}_2(p)$ which minimizes (7.3) (for $\eta = 0$). It may be found from equation (7.5) (without the last term on the left side). The solution of this equation is

$$\hat{f}_2 = \hat{f}_3\hat{f}_1^{\dagger}\left(\hat{f}_1\hat{f}_1^{\dagger} + \left(\epsilon + \nu p^2\right)\mathbf{1}\right)^{-1}, \tag{7.7}$$

where $\mathbf{1}$ is a unit matrix.

Only one ($s = 0$) block is nonzero. In this 9×9 block only one matrix element is nonzero:

$$\left(\hat{f}_3\hat{f}_1^{\dagger}\right)_{31} = -i\frac{64\pi^2 p}{3\sqrt{3}\left(1+p^2\right)^4}.$$

The $s = 0$ block of $(\hat{f}_1\hat{f}_1^{\dagger} + (\epsilon + \nu p^2)\mathbf{1})^{-1}$ is the 9×9 diagonal matrix

$$\begin{bmatrix} \frac{1}{\frac{64\pi^2}{(1+p^2)^4}+\epsilon+\nu p^2} & 0 & 0 \\ 0 & \frac{1}{\epsilon+\nu p^2} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{\epsilon+\nu p^2} \end{bmatrix}. \tag{7.8}$$

Therefore, $\hat{f}_2(p)$ is given by the 9×9 matrix with only one nonzero element

$$\left(\hat{f}_2\right)_{31}(p) = -i\frac{64\pi^2 p}{3\sqrt{3}\left(\epsilon\left(1+p^2\right)^4 + \nu p^2\left(1+p^2\right)^4 + 64\pi^2\right)}.$$

The inverse Fourier transform gives the solution

$$
\begin{aligned}
f_2(\mathbf{r}, A) &= \frac{1}{2\pi^2} \int_0^{\infty} \left(\hat{f}_2\right)_{31}(p)\, U^0_{0,0;1,0}(\mathbf{r}, A; p) p^2\, dp \\
&= (\cos\beta \cos\theta + \sin\beta \sin\theta \cos(\phi - \beta))\ \tilde{f}_2(r, \epsilon, \nu)
\end{aligned}
$$

where

$$
\tilde{f}_2(r, \epsilon, \nu) = \int_0^{\infty} \frac{32p\,(\sin(pr) - pr\, \cos(pr))}{3r^2\left(\epsilon\left(1+p^2\right)^4 + \nu p^2\left(1+p^2\right)^4 + 64\pi^2\right)}\, dp\,. \tag{7.9}
$$

Again, α, β, γ are the $z-x-z$ Euler angles, and θ, ϕ are polar and azimuthal angles of the translation vector $\mathbf{r}$.

We now discuss the choice of small parameters ϵ and ν. After substitution of the solution $\hat{f}_2(p)$ into the functional (7.4) the ϵ, ν dependent part of the functional may be written as

$$
C(\epsilon, \nu) = \int_0^{\infty} \frac{32p^2\left(\epsilon\left(1+p^2\right)^2 + \nu p^2\left(1+p^2\right)^2\right)^2 + 2048\,\epsilon\,\pi^2 p^2 + 2048\,\nu\,\pi^2 p^4}{27\left(\epsilon\left(1+p^2\right)^4 + \nu p^2\left(1+p^2\right)^4 + 64\pi^2\right)^2}\, p^2\, dp\,,
$$

where we dropped the terms which do not depend on ϵ, ν. We depict $C(\epsilon, \nu)$ in Figure 3. $C(\epsilon, \nu)$ does not have any global minimum except $\epsilon = 0$, $\nu = 0$. However, we cannot choose the parameters ϵ and ν as small as we wish because for each value of ϵ we can find an approximate value of ν where the solution starts to develop "unpleasant" behavior. For $\epsilon = 0$ we depict in Figure 4 the values of $\tilde{f}_2(r, 0, \nu)$ (we depict the range of values of $\tilde{f}_2$ between -0.2 and 0.4) as a function of r, ν. We see that for values of ν smaller than $10^{-3} - 10^{-4}$ the solution is increasing rapidly at $r \approx 0.1 - 0.3$ (and exceeds the "natural" scale≈ 1 of maximal values of $f_1(g)$ and $f_3(g)$) and oscillatory "tails" appear at $r \approx 1.0 - 2.5$. The ν dependence of $\tilde{f}_2$ is depicted in Figure 5a for $r = 0.25$, and in Figure 5b for $r = 1$. Thus, a value of ν in the range $10^{-4} - 10^{-3}$ is an appropriate choice of ν for $\epsilon = 0$. The "threshold" solution $\tilde{f}_2$ is depicted in Figure 6 for $\nu = 0.001$ and $\epsilon = 0$.

We may also find the "threshold" value of ϵ for $\nu = 0$ (an analogous analysis shows that ϵ should be chosen in the range $\epsilon_t \approx 5 \cdot 10^{-3} - 5 \cdot 10^{-2}$) and we may find ν for any ϵ between 0 and ϵ_t. The particular choice of ϵ and ν depends on whether we want to emphasis the restriction on absolute value of $f_2(g)$ (parameter ϵ) or pay more attention to the restriction on magnitude of derivatives of the function (parameter ν).

We note that this example was chosen because the regularization technique can be demonstrated in closed form. We have implemented this technique for the design of planar robot arms [22]. Good results are achieved (in the sense of least-squared error) for 9×9 truncated Fourier matrices in the planar case. To achieve similar accuracy in the three-dimensional case we would expect to use up to $l = 4$ and $|s| = 4$ Fourier matrices (matrices truncated at $l = 4$ are 25×25 for $s = 0$), and use a six dimensional grid for discrete values of translations and rotations. Implementing the theory presented in this article in manipulator design thus remains computationally challenging, and we are actively addressing this problem using the techniques presented here.

7.3 A Note About Real Solutions

In the above example we find a real solution for given real functions $f_1(g)$ and $f_3(g)$. This is a particular example of the fact that if $f_1(g)$ and $f_3(g)$ are real, then $f_2(g)$ given by (7.2) (for non-singular $\hat{f}_1(p)$) or (7.7) (in the regularized problem) is a real function. It follows from the following consideration. Even after truncating the Fourier transform matrices of $f_1(g)$ and $f_3(g)$ the approximation to $f_1(g)$ and $f_3(g)$ given by inverse Fourier transform equation (4.4) (where the

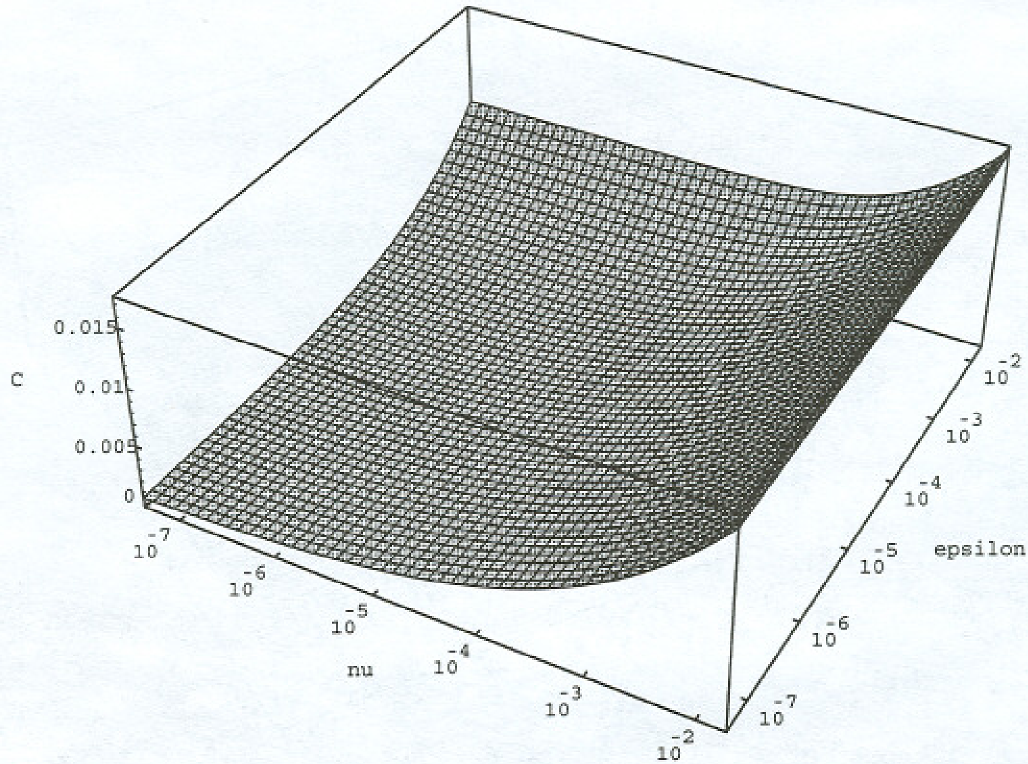

FIGURE 3 The $C(\epsilon, \nu)$ dependence on ϵ, ν parameters. The ϵ, ν axes are depicted in the logarithmic scale.

sum is for l, l' from 0 to some l_{chop} and for s from $-l_{chop}$ to l_{chop}) gives real functions. This is valid because the matrix elements of the Fourier transform and the matrix elements of $U(g, p)$ satisfy (4.8) and (3.16), which means that

$$\overline{\hat{f}^{s}_{l',0;l,0}\, U^{s}_{l,0;l',0}} = \hat{f}^{s}_{l',0;l,0}\, U^{s}_{l,0;l',0}$$

and

$$\overline{\hat{f}^{s}_{l',m';l,m}\, U^{s}_{l,m;l',m'}} = \hat{f}^{s}_{l',-m';l,-m}\, U^{s}_{l,-m;l',-m'}$$

where m or $m' \neq 0$, and $\hat{f}$ can be either $\hat{f}_1(p)$ or $\hat{f}_3(p)$. This means that the sum in (4.4) is real because $(l', 0; l, 0)$ elements are real and $(l', m'; l, m)$ elements come in pairs with the complex conjugate elements $(l', -m'; l, -m)$.

If the truncated $\hat{f}_1(p)$ is invertible, then the solution of the convolution equation (1.1) exists (where $f_1(g)$ and $f_3(g)$ are approximated by the finite sum in (4.4)). If $f_2(g)$ has imaginary part $f_2^{im}(g)$ then $(f_1 * f_2^{im})(g) = 0$, because $f_1(g)$ and $f_3(g)$ are real functions. But this equation means that $\hat{f}_2^{im}(p)\hat{f}_1(p) = 0$ which is only possible for $\hat{f}_2^{im}(p) = 0$ for non-singular $\hat{f}_1(p)$. It means that the imaginary contribution is zero. We may conclude also that if two (or several) matrices satisfy the symmetry relations (4.8) the product of these matrices (or inverse matrices) also satisfy the symmetry relations (4.8) (it follows from the above fact and the fact that if $f_1(g)$ and $f_3(g)$ are real, the convolution of these functions is also real).

If the $\hat{f}_1(p)$ is a singular matrix then the regularized solution given by (7.7) (or solution of (7.4)) is a real solution after taking the Fourier transform. It follows from the above mentioned fact that the product of several matrices with the symmetry properties (4.8) has the same symmetry [it is clear that the matrices proportional to the unit matrix and matrix A have the symmetry (4.8)]. Or we may conclude that only a real solution minimizes the functional (7.3), because if we conclude that

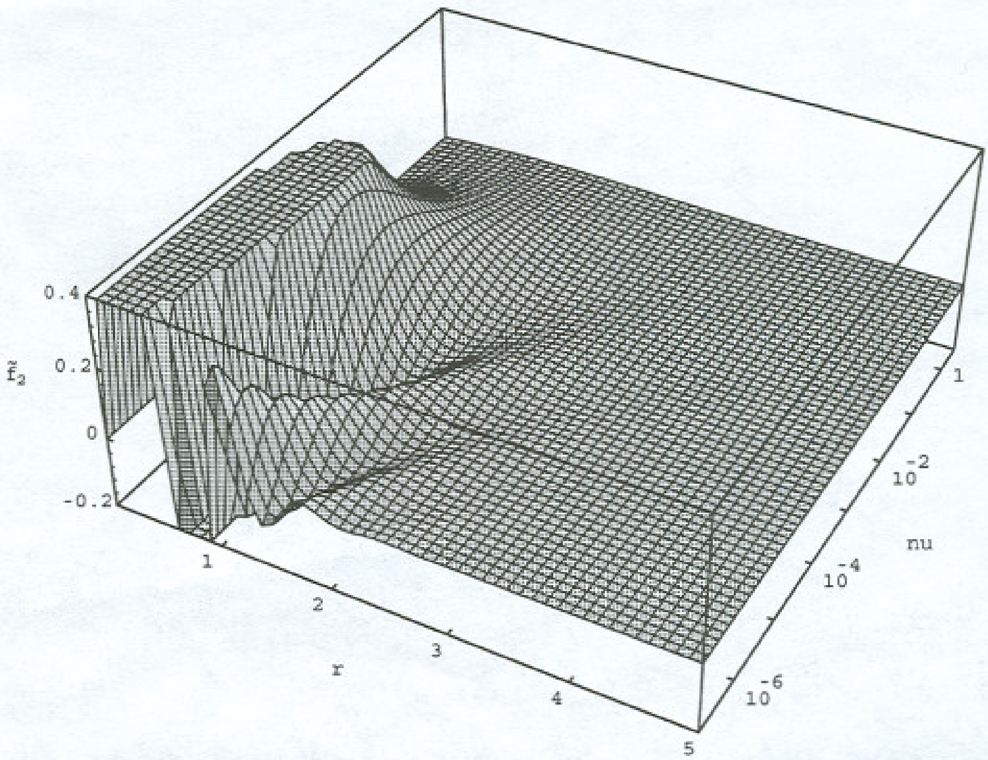

FIGURE 4 The r, ν-dependence of $\tilde{f}_2(r, 0, \nu)$. The ν axis is depicted in the logarithmic scale.

some complex function $f_2(g)$ minimizes (7.3) we find that we may decrease further the value of this functional (for real $f_1(g)$ and $f_3(g)$) if we put the imaginary part of $f_2(g)$ to be zero.

8. Solving PDEs from Polymer Science

By defining

$$u\left(\tilde{X}_i, p\right) \triangleq \frac{d}{dt}\left(U\left(\exp\left(t\tilde{X}_i\right), p\right)\Big|_{t=0}, \tag{8.1}$$

making a change of variables, and using the fact that $SE(3)$ is unimodular, it may be shown that [7]:

$$\mathcal{F}\left(\tilde{X}_i^R f\right) = u\left(\tilde{X}_i, p\right)\hat{f}(p) .$$

This means that (2.1) can be transformed to the infinite system of linear differential equations:

$$\frac{d\hat{f}^s}{dL} = B^s \hat{f}^s , \tag{8.2}$$

where

$$B^s = \frac{1}{2}\sum_{i,j=1}^{3} D_{ij} u^s\left(\tilde{X}_i, p\right) u^s\left(\tilde{X}_j, p\right) + \sum_{i=1}^{3} d_i u^s\left(\tilde{X}_i, p\right) - u^s\left(\tilde{X}_6, p\right) .$$

In principle, $f(\mathbf{a}, A; L)$ is then found by simply substituting

$$\hat{f}^s(p; L) = \exp\left(LB^s\right)$$

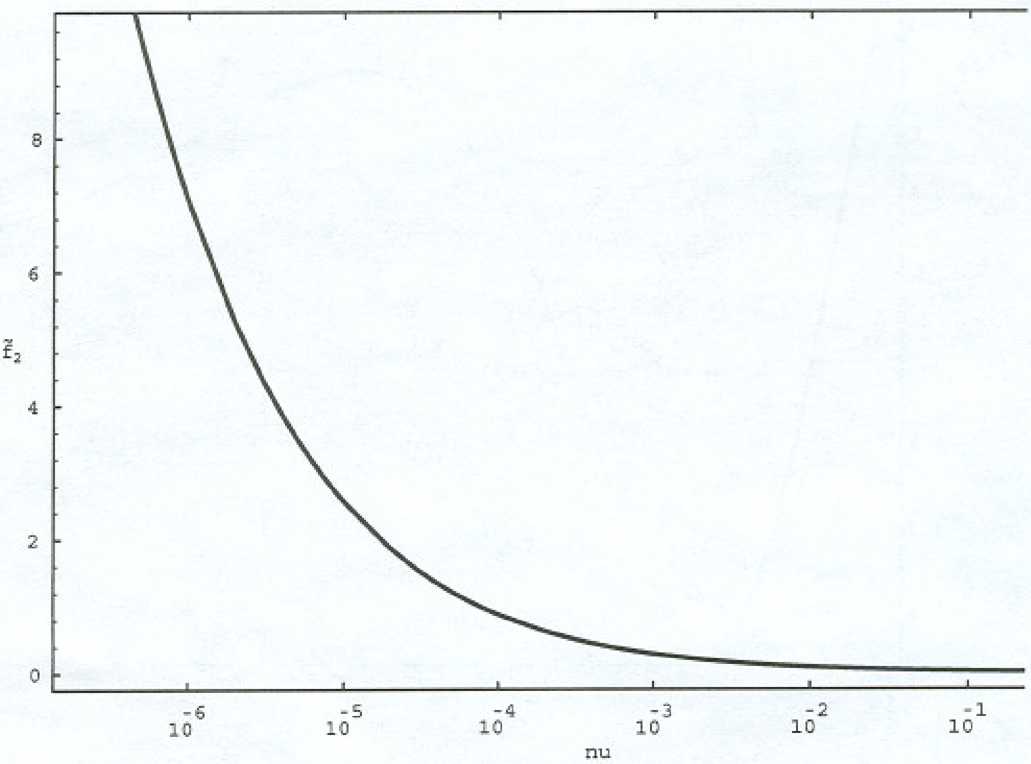

FIGURE 5 (a) The ν dependence of $\tilde{f}_2(0.25, 0, \nu)$. The ν axis is depicted in the logarithmic scale.

into the $SE(3)$ Fourier inversion formula (4.4). Explicitly, for $i = 1, 2, 3$ we have

$$u^s\left(\tilde{X}_i, p\right) = \frac{d}{dt} U^s_{l',m';l,m}\left(\mathbf{0},\ \mathrm{ROT}\left(\mathbf{e}_i, t\right); p\right)|_{t=0} = \delta_{l,l'} \frac{d}{dt} U^l_{m',m}\left(\mathrm{ROT}\left(\mathbf{e}_i, t\right)\right)|_{t=0}$$

where $\mathrm{ROT}(\mathbf{e}_i, \theta)$ is a counterclockwise rotatation by θ round natural basis vector $\mathbf{e}_i$ (i.e., $(\mathbf{e}_i)_j = \delta_{ij}$).

The second equality in the above expression for $u^s(\tilde{X}_i, p)$ follows easily from the structure of the matrix elements $U^s_{l',m';l,m}$. We note the $SO(3)$ operational property

$$\frac{d}{dt} U^l_{mn}\left(\mathrm{ROT}\left(\mathbf{e}_1, t\right)\right)|_{t=0} = \frac{1}{2} c^l_{-n} \delta_{m+1,n} - \frac{1}{2} c^l_n \delta_{m-1,n} \tag{8.3}$$

$$\frac{d}{dt} U^l_{mn}\left(\mathrm{ROT}\left(\mathbf{e}_2, t\right)\right)|_{t=0} = \frac{i}{2} c^l_{-n} \delta_{m+1,n} + \frac{i}{2} c^l_n \delta_{m-1,n} \tag{8.4}$$

$$\frac{d}{dt} U^l_{mn}\left(\mathrm{ROT}\left(\mathbf{e}_3, t\right)\right)|_{t=0} = -in\delta_{m,n} \tag{8.5}$$

where $c^l_n = \sqrt{(l-n)(l+n+1)}$ for $l \geq |n|$. From the operational property (5.5) we see that

$$\begin{aligned} u^s_{l',m';l,m}\left(\tilde{X}_6, p\right) &= \frac{d}{dt} U^s_{l',m';l,m}\left(t\mathbf{e}_3, I; p\right)|_{t=0} \\ &= ip\kappa^s_{l',m'} \delta_{l'-1,l} \delta_{m',m} + ip \frac{m' s}{l'(l'+1)} \delta_{l'l} \delta_{m',m} + ip\, \kappa^s_{l,m}\, \delta_{l',l-1} \delta_{m',m} \end{aligned} \tag{8.6}$$

where

$$\kappa^s_{l',m'} = \left(\frac{\left(l'^2 - m'^2\right)\left(l'^2 - s^2\right)}{(2l'+1)(2l'-1)\, l'^2}\right)^{1/2}.$$

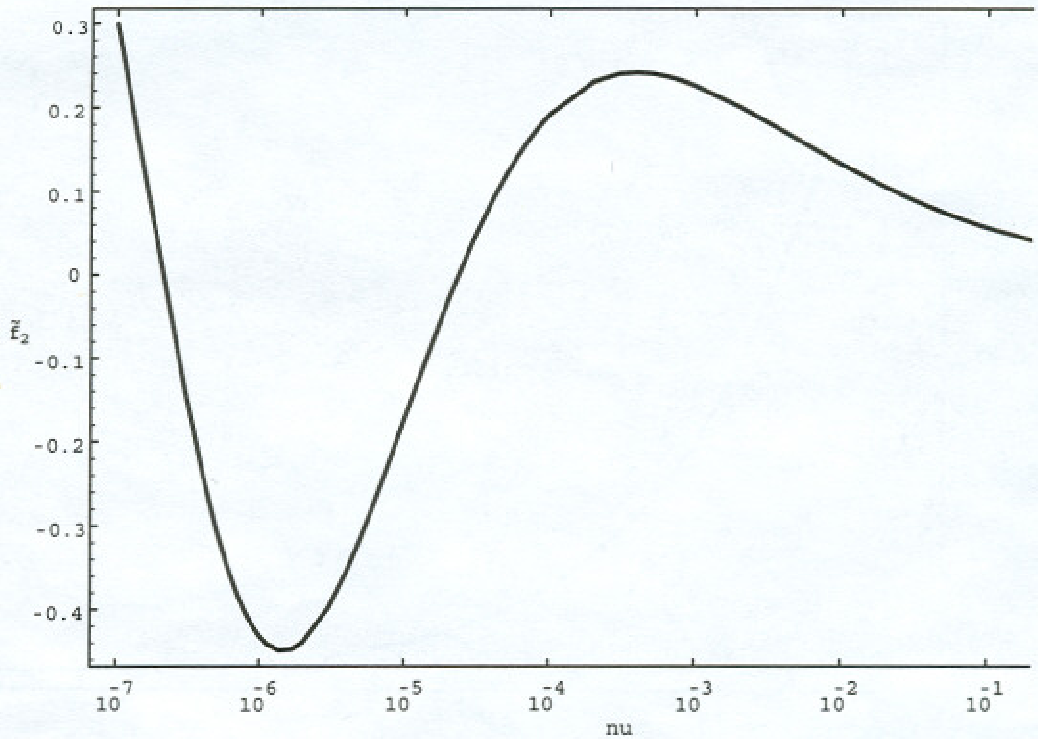

FIGURE 5 (b) The ν dependence of $\tilde{f}_2(1, 0, \nu)$.

This allows us to write the elements of $B^s(p)$ as

$$B^s_{l',m';l,m} = K^l_{m',m}\delta_{l',l} - ip\kappa^s_{l',m'}\delta_{l'-1,l}\delta_{m',m} - ip\,\frac{m'\,s}{l'\,(l'+1)}\,\delta_{l'l}\delta_{m',m} - ip\,\kappa^s_{l,m}\,\delta_{l',l-1}\delta_{m',m}$$

where

$$\begin{aligned} K^l_{m',m} &= \frac{1}{2}\sum_{i,j=1}^{3} D_{ij}\sum_{n=-l}^{l}\left(\frac{d}{dt}U^l_{m',n}\,(\mathrm{ROT}\,(\mathbf{e}_i,t))|_{t=0}\right)\left(\frac{d}{dt}U^l_{n,m}\,\left(\mathrm{ROT}\,(\mathbf{e}_j,t)\right)\Big|_{t=0}\right) \\ &\quad + \sum_{i=1}^{3} d_i\left(\frac{d}{dt}U^l_{m',m}\,(\mathrm{ROT}\,(\mathbf{e}_i,t))|_{t=0}\right). \end{aligned}$$

9. Conclusions

In this article we reviewed the theory of irreducible unitary representations of the Euclidean motion group in three-dimensional space. We calculated the matrix elements of these irreducible unitary representations and used this set of matrix elements to define the Fourier transform of functions on the motion group. We derived new symmetry and operational properties of this Fourier transform. These tools were applied to the solution of the convolution equation (1.1), and the regularized problem (7.3) was explicitly solved for particular functions. We discussed the effect of the choice of small parameters in the regularized problem (7.3). The operation calculus developed here for the Euclidean motion group was also used in this article to solve a class of partial differential equations from theoretical polymer science.

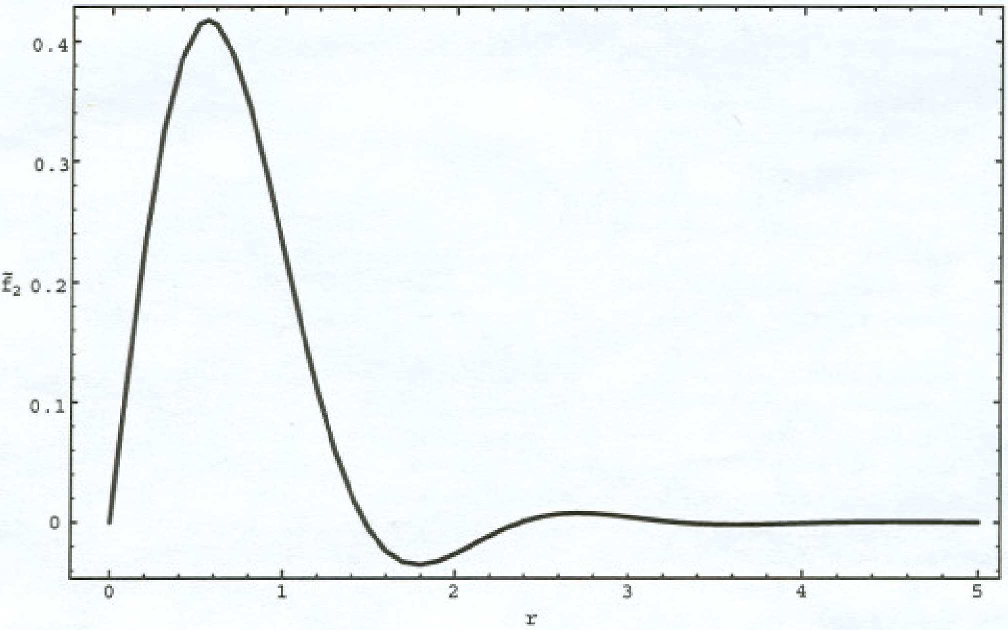

FIGURE 6 The r dependence of $\bar{f}_2$ for $\nu = 0.001$ and $\epsilon = 0$.

This article illustrates that the non-Abelian Fourier transform is a powerful tool for the solution of linear integral equations and partial differential equations on the motion group.

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