Proceedings of
A Symposium Commemorating the
Legacy, Works, and Life of
Sir Robert Stawell Ball
Upon the 100th Anniversary of
A Treatise on the Theory of Screws
July 9-11, 2000
University of Cambridge, Trinity College

Fourier Analysis on Motion Using Screw Parameters

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Abstract

In this paper, the author presents the bi-invariant (Haar) integral for the group of rigid-body motions (Euclidean group) in three-dimensional space in terms of finite screw parameters, and proceeds to develop the matrix elements of irreducible unitary representations in this parameterization. This allows one to integrate and expand functions of motion described in terms of screw parameters.

In robot kinematics, the screw-parameter description of a finite rigid-body motion is well known. This description of motion in three-dimensional space (which follows from Ball's work on finite screws) provides an elegant way to view rigid-body kinematics. In contrast, the theoretical physics community usually uses Euler angles and spherical coordinates to parameterize rigid-body motions. It therefore comes as no surprise that in the field of Fourier analysis on groups, which has been developed in large part by theoretical physicists, that the Euler-angle/spherical coordinate description of rigid-body motions is most common.

The contribution of this paper is to formulate Fourier analysis on the group of rigid-body motions in terms of screw parameters. The geometrically meaningful nature of the screw parameters combined with the group Fourier transform provides a tool for insight into problems that can be posed as convolutions on the Euclidean group. Such problems include workspace generation of serial linkages, kinematic error propagation, and the statistical mechanics of macromolecules.

1 A Review of Screw Theory

The Euclidean motion group (also called the special Euclidean group of 3-dimensional space and denoted $SE(3)$) consists of all pairs of the form $g = (\mathbf{a}, A)$ where $A \in SO(3)$ and $\mathbf{a} \in I\!R^3$ (Recall that $SO(3)$ denotes the set of all 3×3 rotation matrices). For any $g = (\mathbf{a}, A)$ and $h = (\mathbf{b}, B) \in SE(3)$, the composition law for rigid-body (Euclidean) motions is written as $g \circ h = (\mathbf{a} + A\mathbf{b}, AB)$, and the inverse of a motion is $g^{-1} = (-A^T\mathbf{a}, A^T)$. Alternately, one may represent any element of $SE(3)$ as an 4×4 homogeneous transformation matrix of the form:

$$H(g) = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{0}^T & 1 \end{pmatrix}.$$

Clearly, $H(g)H(h) = H(g \circ h)$ and $H(g^{-1}) = H^{-1}(g)$ and the mapping $g \leftrightarrow H(g)$ between $SE(3)$ and the set of homogeneous transformation matrices is bijective. In group-theoretic language, $SE(3)$ is the semidirect product of $I\!R^3$ with the special orthogonal group, $SO(3)$, and we write $SE(3) = I\!R^3 \lhd_\varphi SO(3)$.

A screw axis is a line in space about which a rotation is performed and along which a translation is performed [1]. Given a motion $(\mathbf{a}, A)$, the direction of the screw axis is the axis of the rotation A, but the translation along the screw axis is not simply $\mathbf{a}$. We now review the decomposition of an arbitrary rigid-body motion into its screw parameters.

Any line in space is specified completely by a direction $\mathbf{n} \in S^2$ and the position of any point $\mathbf{r}$ on the line. Hence, a line is parametrized as

$$\mathbf{L}(t) = \mathbf{r} + t\mathbf{n}, \quad \forall \quad t \in I\!R.$$

Since there are an infinite number of vectors $\mathbf{r}$ on the line that can be chosen, the one which is "most natural" is that which has the smallest magnitude. This is the vector originating at the origin of the coordinate system and terminating at the line to which it intersects orthogonally. Hence the condition $\mathbf{r} \cdot \mathbf{n} = 0$ is satisfied. Since $\mathbf{n}$ is a unit vector and $\mathbf{r}$ satisfies a constraint equation, a line is uniquely specified by only four parameters. Often instead of the pair of line coordinates $(\mathbf{n}, \mathbf{r})$, the pair $(\mathbf{n}, \mathbf{r} \times \mathbf{n})$ is used to describe a line because this implicitly incorporates the constraint $\mathbf{r} \cdot \mathbf{n} = 0$. That is, when $\mathbf{r} \cdot \mathbf{n} = 0$, $\mathbf{r}$ can be reconstructed as $\mathbf{r} = \mathbf{n} \times (\mathbf{r} \times \mathbf{n})$, and it is clear that for unit $\mathbf{n}$, that the pair $(\mathbf{n}, \mathbf{r} \times \mathbf{n})$ has four degrees of freedom. Such a description of lines is called the Plücker coordinates. For more on this subject, and kinematics in general, see [1, 2, 3, 4, 12, 15, 19, 20].

[1] The theory of screws was developed by Sir Robert Stawell Ball (1840-1913) [1].

Given an arbitrary point $\mathbf{x}$ in a rigid body, the transformed position of the same point after translation by d units along a screw axis with direction specified by $\mathbf{n}$ is $\mathbf{x}^{'} = \mathbf{x} + d\mathbf{n}$. Rotation about the same screw axis is given as $\mathbf{x}^{''} = \mathbf{r} + e^{\theta N}(\mathbf{x}^{'} - \mathbf{r})$ where $N = -N^T$ is the unique skew-symmetric matrix with the property $N\mathbf{x} = \mathbf{n} \times \mathbf{x}$.

Since $e^{\theta N}\mathbf{n} = \mathbf{n}$, it does not matter if translation along a screw axis is performed before or after rotation. Either way, $\mathbf{x}^{''} = \mathbf{r} + e^{\theta N}(\mathbf{x} - \mathbf{r}) + d\mathbf{n}$.

Another way to view this is that the homogeneous transforms

$$\mathrm{trans}(\mathbf{n}, d) = \begin{pmatrix} \mathbb{1} & d\mathbf{n} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

and

$$\mathrm{rot}(\mathbf{n}, \mathbf{r}, \theta) = \begin{pmatrix} e^{\theta N} & (\mathbb{1} - \mathrm{e}^{\theta \mathrm{N}})\mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

commute, and the homogeneous transform for a general rigid-body motion along screw axis $(\mathbf{n}, \mathbf{r})$ is given as

$$\mathrm{rot}(\mathbf{n}, \mathbf{r}, \theta)\mathrm{trans}(\mathbf{n}, d) = \mathrm{trans}(\mathbf{n}, d)\mathrm{rot}(\mathbf{n}, \mathbf{r}, \theta) = \begin{pmatrix} e^{\theta N} & (\mathbb{1} - \mathrm{e}^{\theta \mathrm{N}})\mathbf{r} + \mathrm{d}\mathbf{n} \\ \mathbf{0}^T & 1 \end{pmatrix}. \quad (1)$$

A natural question to ask at this point is how the screw axis parameters $(\mathbf{n}, \mathbf{r})$ and motion parameters (θ, d) can be extracted from a given rigid displacement $(\mathbf{a}, A)$. Since this is well known for pure rotations (see e.g., [17]), half the problem is already solved, i.e., $\mathbf{n}$ and θ are calculated by inverting the expression

$$A = A(\mathbf{n}, \theta) = e^{\theta N} = \mathbb{1} + \sin\theta\,\mathrm{N} + (1 - \cos\theta)\mathrm{N}^2 \quad (2)$$

where $\mathbf{n} \in S^2$ and $\theta \in [-\pi, \pi]$.

What remains is to find for given A, $\mathbf{n}$, and $\mathbf{a}$ the variables $\mathbf{r}$ and d satisfying

$$(\mathbb{1} - \mathrm{A})\mathbf{r} + \mathrm{d}\mathbf{n} = \mathbf{a} \quad \text{and} \quad \mathbf{r} \cdot \mathbf{n} = 0.$$

This is achieved by first taking the dot product of the left of the above equations with $\mathbf{n}$ and observing that $\mathbf{n} \cdot e^{\theta N}\mathbf{r} = \mathbf{n} \cdot \mathbf{r} = 0$, and so

$$d = \mathbf{a} \cdot \mathbf{n}.$$

We then can write

$$(\mathbb{1} - \mathrm{e}^{\theta \mathrm{N}})\mathbf{r} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{n})\mathbf{n}. \quad (3)$$

Next we introduce the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{n_1^2 + n_2^2}} \begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix}, \quad (4)$$

where n_1 and n_2 are the first two components of $\mathbf{n}$. This is simply a choice we make that has the property $\mathbf{u} \cdot \mathbf{n} = 0$ and $\|\mathbf{u}\| = 1$. In this way, $\{\mathbf{n}, \mathbf{u}, \mathbf{n} \times \mathbf{u}\}$ forms a right-handed coordinate system. We may then write $\mathbf{r}$ in the form

$$\mathbf{r} = b\mathbf{u} + c(\mathbf{n} \times \mathbf{u}).$$

This may be substituted into (3) and projected onto $\mathbf{u}$ and $\mathbf{n} \times \mathbf{u}$ to yield the set of linear equations

$$\begin{pmatrix} 1 - \cos\theta & \sin\theta \\ -\sin\theta & 1 - \cos\theta \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} \mathbf{a} \cdot \mathbf{u} \\ \mathbf{a} \cdot (\mathbf{n} \times \mathbf{u}) \end{pmatrix}, \tag{5}$$

which is solved for b and c, provided $\theta \neq 0$. In this special case, any $\mathbf{r}$ in the plane normal to $\mathbf{n}$ will do.

We note also that the translation vector $\mathbf{a}$ can be expressed as

$$\mathbf{a} = [c\sin\theta + b(1 - \cos\theta)]\mathbf{u} + [-b\sin\theta + c(1 - \cos\theta)](\mathbf{n} \times \mathbf{u}) + d\mathbf{n}. \tag{6}$$

This means that the magnitude of the translation can be written in terms of screw parameters as

$$a = \|\mathbf{a}\| = \sqrt{(b^2 + c^2)\sin^2\theta + (b^2 + c^2)(1 - \cos\theta)^2 + d^2}.$$

The bi-invariant volume element (Haar measure) with which to integrate functions on $SE(3)$ is (to within an arbitrary constant) of the form

$$dg = dA d\mathbf{a}$$

when $g = (A, \mathbf{a})$, where dA is the normalized Haar measure for $SO(3)$ and $d\mathbf{a} = da_1 da_2 da_3$ is the Haar measure for $I\!R^3$. Using the screw parameterization in (6), we see that

$$d\mathbf{a} = 4\sin^2(\theta/2) db dc dd.$$

Likewise, to within a constant factor,

$$dA = 4\sin^2(\theta/2) d\mathbf{n}$$

where $d\mathbf{n}$ is an integration measure for the sphere. For instance, if $\mathbf{n} = \mathbf{n}(\lambda, \nu)$ where λ and ν are respectively the polar and azimuthal angles, then

$$d\mathbf{n} = \sin\lambda d\lambda d\nu.$$

2 Group Representation Theory

A group representation is a mapping from the group into a set of invertible linear transformations (matrices) that preserves the group law. In kinematics, the description of rigid-body motions using homogeneous transformation matrices is an example of a representation of $SE(3)$. More generally, a goal of representation theory is the enumeration of all representation matrices that cannot be block diagonalized by similarity transformation. Such representations are called *irreducible*. It may be shown that every irreducible representation is equivalent under similarity transformation to a unitary matrix. Since unitary matrices are convenient to work with (since their inverse is the Hermitian conjugate), a powerful mathematical theory has been built on irreducible unitary representations (IURs) of groups.

Basically, the collection of matrix elements of all the IURs of a group serves as an orthogonal basis for the set of all square-integrable functions on the group. That is, if G is a group and $g \in G$, then any function $f \in L^2(G)$ can be expanded as a weighted sum (or integral) of $U^l_{mn}(g)$ over all m, n, l. Here m, n indicate the entry (element) in the unitary matrix $U^l(g)$. The parameter l enumerates the set of IURs. In the past, the author and coworkers have used the IURs of $SO(3)$ in the commutation of a spherical motor [5], and the IURs of $SE(2)$ (the group of motions of the plane) in the design and inverse kinematics of discretely-actuated manipulator arms [11, 6].

We now review the form which the matrix elements $\tilde{U}^l_{mn}(A)$ take when rotations are parameterized as $A(\theta, \mathbf{n}(\lambda, \nu)) = \exp(\theta N(\lambda, \nu))$ where $N\mathbf{x} = \mathbf{n} \times \mathbf{x}$, and $\mathbf{n}$ is a unit vector defining the axis of rotation with spherical coordinates (λ, ν).

One finds the matrix to be of the form [21]

$$\tilde{U}^l_{mn}(A(\theta, \mathbf{n}(\lambda, \nu))) = i^{m-n} e^{-i(m-n)\nu} \left(\frac{1 - i \tan\theta/2 \cos\lambda}{\sqrt{1 + \tan^2\theta/2 \cos^2\lambda}} \right)^{m+n} P^l_{mn}(x)$$

where x satisfies

$$\sin x/2 = \sin\theta/2 \sin\lambda$$

and $P^l_{mn}(x)$ are polynomials defined in [6]. The range of indices is $l = 0, 1, 2, ...$ and $m, n \in \{-l, -l+1, ..., l-1, l\}$. A function with band-limit B in this context is one for which all $l \geq B$ can be neglected in the expansion.

3 Irreducible Unitary Representations of $SE(3)$

We define unitary representations of $SE(3)$ (see e.g. [6, 16, 18, 22] for discussions and definitions) in the following way.

We start to construct the representation of the motion group in the space of functions $\varphi(\mathbf{p}) \in \mathcal{L}^2(\hat{\mathbf{T}})$, where $\hat{T}$ is a dual (frequency) space of the $I\!R^3$ subgroup. Functions $\varphi(\mathbf{p})$ correspond to the Fourier transforms of the functions $\phi(\mathbf{r}) \in \mathcal{L}^2(T)$, where $T = I\!R^3$, are defined as

$$\varphi(\mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int_T e^{-i\mathbf{p}\cdot\mathbf{r}} \phi(\mathbf{r})\, d\mathbf{r}. \tag{7}$$

The rotation subgroup $SO(3)$ of the motion group acts on $\hat{T}$ by rotations, so $\hat{T}$ is divided into orbits S_p, where S_p are S^2 spheres of radius $p = \mid \mathbf{p} \mid$. The translation operator acts on $\varphi(\mathbf{p})$ as

$$(U(\mathbf{a}, \mathbb{1})\varphi)(\mathbf{p}) = \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot\mathbf{a}}\, \varphi(\mathbf{p}). \tag{8}$$

Therefore, the irreducible representations of the motion group may be built on spaces $\varphi(\mathbf{p}) \in \mathcal{L}^2(S_p)$, with the inner product defined as

$$(\varphi_1, \varphi_2) = \int_{\lambda_p=0}^{\pi} \int_{\nu_p=0}^{2\pi} \overline{\varphi_1(\mathbf{p})}\, \varphi_2(\mathbf{p}) \sin\lambda_p \, d\lambda_p \, d\nu_p \ , \tag{9}$$

where $\mathbf{p} = (p \sin\lambda_p \cos\nu_p,\ p \sin\lambda_p \sin\nu_p,\ p \cos\lambda_p)$, and $p > 0$, $0 \leq \lambda_p \leq \pi$, $0 \leq \nu_p \leq 2\pi$.

The inner product (φ_1, φ_2) is invariant with respect to transformations

$$\varphi(\mathbf{p}) \rightarrow e^{i\alpha}\, \varphi(A^{-1}\, \mathbf{p}) \ , \tag{10}$$

where $A \in SO(3)$ and $0 \leq \alpha \leq 2\pi$.

The parameter α in (10) may, in general, depend on p and group element $A \in SO(3)$. In this case, different functions $\alpha_s(p, A)$ (where s enumerates the irreducible representations of $SO(2)$), which are nonlinear functions of group element A, correspond to different irreducible representations of the motion group. Functions $\varphi(\mathbf{p})$, thus, may have different *internal* properties with respect to rotations.

With the help of functions $\alpha_s(p, A)$ we may construct the representations of $G = SE(3) \simeq \hat{T} \lhd_\varphi SO(3)$ from representations of its subgroup $G' = \hat{T} \lhd_\varphi SO(2)$ using the method of induced representations. In our case (we disregard for the moment the translation group $\hat{T}$), $G = SO(3)$, $H = SO(2)$ and $\sigma = \mathbf{p} \in S_p \simeq SO(3)/SO(2)$.

To construct the representations of the motion group explicitly, we choose a particular vector $\hat{\mathbf{p}} = (0, 0, p)$ on each orbit S_p. The vector $\hat{\mathbf{p}}$ is invariant with respect to rotations from the $SO(2)$ subgroup of $SO(3)$

$$\Lambda\, \hat{\mathbf{p}} = \hat{\mathbf{p}}; \qquad \Lambda \in H_{\hat{\mathbf{p}}} = SO(2), \tag{11}$$

where $H_{\hat{\mathbf{p}}}$ is a little group of $\hat{\mathbf{p}}$. For each $\mathbf{p} \in S_p$ we may find $R_{\mathbf{p}} \in SO(3)/SO(2)$, such that

$$R_{\mathbf{p}}\, \hat{\mathbf{p}} = \mathbf{p}.$$

Then for any $A \in SO(3)$, one may check that

$$(R_{\mathbf{p}}^{-1}\, A\, R_{A^{-1}\mathbf{p}})\, \hat{\mathbf{p}} = \hat{\mathbf{p}}.$$

Therefore, $Q(\mathbf{p}, A) = (R_{\mathbf{p}}^{-1}\, A\, R_{A^{-1}\mathbf{p}}) \in H_{\hat{\mathbf{p}}}$. The representations of $H_{\hat{\mathbf{p}}}$ may be taken to be of the form

$$\Delta_s : \quad \phi \rightarrow e^{is\phi} \ , \ 0 \leq \phi \leq 2\pi$$

for $s = 0, \pm 1, \pm 2, ...$

Thus, we may construct the induced representation $(\hat{T} \lhd_{\varphi} \Delta_s(H_{\hat{\mathbf{p}}})) \uparrow SE(3)$ of the motion group from the representations of its subgroup $\hat{T} \lhd_{\varphi} H_{\hat{\mathbf{p}}}$.

Definition. [6] *The unitary representations $U^s(\mathbf{a}, A)$ of $SE(3)$, which act on the space of functions $\varphi(\mathbf{p})$ with the inner product (9), are defined by*

$$(U^s(\mathbf{a}, A)\varphi)(\mathbf{p}) = e^{-i\mathbf{p}\cdot\mathbf{a}}\, \Delta_s(R_{\mathbf{p}}^{-1}\, A\, R_{A^{-1}\mathbf{p}})\, \varphi(A^{-1}\mathbf{p}) \quad , \tag{12}$$

where $A \in SO(3)$, Δ_s are representations of $H_{\hat{\mathbf{p}}}$ and $s = 0, \pm 1, \pm 2, ...$

Each representation characterized by $p = \|\mathbf{p}\|$ and s is irreducible (they, however, become reducible if we restrict $SE(3)$ to $SO(3)$, i.e. when $a = \|\mathbf{a}\| = 0$). They are unitary because $(U^s(\mathbf{a}, A)\varphi_1, U^s(\mathbf{a}, A)\varphi_2) = (\varphi_1, \varphi_2)$.

Representations (12), which we denote below by $U^s(g, p)$, satisfy the homomorphism properties

$$U^s(g_1 \circ g_2, p) = U^s(g_1, p) \cdot U^s(g_2, p) \quad ,$$

where $\circ$ is the group operation. The corresponding multiplication law for the $Q(\mathbf{p}, A)$ factors is [23]

$$Q(\mathbf{p}, A)\, Q(A^{-1}\mathbf{p}, A^{-1}B) = Q(\mathbf{p}, B). \tag{13}$$

4 Matrix elements

To obtain the matrix elements of the unitary representations, we use the group property

$$U^s(\mathbf{a}, A) = U^s(\mathbf{a}, \mathbb{1}) \cdot \mathrm{U^s(0, A)} \tag{14}$$

The basis eigenfunctions of the irreducible representations (12) of $SE(3)$ may be enumerated by the integer numbers l, m (for each s and p). We note that the values $l(l+1), m, ps$ and $-p^2$ correspond to the eigenvalues of the generators $\mathbf{J^2}, J^3, \mathbf{P}\cdot\mathbf{J}, \mathbf{P}\cdot\mathbf{P}$

(where $J^i, P^i,\ i = 1,2,3$ are generators of rotation and translation) of the Lie algebra $se(3)$ (see [6, 16]) of the motion group $SE(3)$, which may be diagonalized simultaneously (i.e. they commute). The restrictions for the l, m, s numbers are $l \geq \mid s \mid;\quad l \geq \mid m \mid$.

The basis functions may be expressed in the form [16]

$$h^l_{m\,s}(\Theta, \Phi) \;=\; Q^l_{s,m}(\cos\,\Theta)\, e^{i(m+s)\Phi} \tag{15}$$

where

$$Q^l_{-s,m}(\cos\,\Theta) \;=\; (-1)^{l-s}\sqrt{\frac{2l+1}{4\pi}}\, P^l_{s\,m}(\cos\,\Theta)\quad , \tag{16}$$

and generalized Legendre polynomials $P^l_{m\,s}(\cos\,\Theta)$ are given as in Vilenkin [22].

It may be shown that the basis functions $h^l_{m\,s}$ are transformed under the rotations $h^l_{m\,s}(\mathbf{p}) \;\rightarrow\; \Delta_s(Q(\mathbf{p}, A))\, h^l_{m\,s}(A^{-1}\mathbf{p})$ as (see [6] Chapter 9 for the proof):

$$(U^s(0, A)\, h^l_{m\,s})(\mathbf{p}) \;=\; \sum_{n=-l}^{l} \tilde{U}^l_{n\,m}(A) h^l_{n\,s}(\mathbf{p})\quad , \tag{17}$$

where the matrix elements $\tilde{U}^l_{n\,m}(A)$ are are defined in the previous section. We note that the rotation matrix elements do not depend on s.

The translation matrix elements are given by the integral [16]

$$\begin{aligned}(h^{l'}_{m'\,s}, U^s(\mathbf{a}, \mathbb{1})\mathrm{h^l_{m\,s}}) \;=\; [\mathrm{l', m' \mid p, s \mid l, m}](\mathbf{a}) \;=\; \\ (4\pi)^{1/2} \sum_{k=\mid l'-l\mid}^{l'+l} i^k \sqrt{\frac{(2l'+1)(2k+1)}{(2l+1)}}\, j_k(p\,a)\, C(k, 0; l', s \mid l, s) \\ \cdot C(k, m-m'; l', m' \mid l, m)\, Y^{m-m'}_k(\Theta, \Phi)\quad ,\end{aligned} \tag{18}$$

where Θ, Φ are polar and azimuthal angles of $\mathbf{a}$, $C(k, m-m'; l', m' \mid l, m)$ are Clebsch-Gordan coefficients (see, for example, [6, 21]).

Finally, using the group property (14), the matrix elements of the unitary representation $U^s(g, p)$ (12) (for $s = 0, \pm 1, \pm 2, ...$) are expressed as

$$U^s_{l',m';l,m}(\mathbf{a}, A; p) \;=\; \sum_{j=-l}^{l} [l', m' \mid p, s \mid l, j](\mathbf{a})\, \tilde{U}^l_{j\,m}(A) \tag{19}$$

Because (18) contains only half-integer Bessel functions, all matrix elements may be expressed in terms of elementary functions. Below we have shown the first matrix elements in explicit form (using the notation $a = \|\mathbf{a}\|$ and $p = \|\mathbf{p}\|$):

$$U^0_{0,0;0,0}(\mathbf{a}, A; p) \;=\; \frac{\sin(a\,p)}{a\,p}\quad .$$

5 The Fourier Transform for $SE(3)$

Here we define the Fourier transform of functions $f(\mathbf{a}, A) \in \mathcal{L}^2(SE(3))$. The inner product of two such functions is given by

$$(f_1, f_2) \;=\; \int_{I\!R^3} \int_{SO(3)} \overline{f_1(\mathbf{a}, A)}\, f_2(\mathbf{a}, A)\, dA\, d^3\mathbf{a} \quad . \tag{20}$$

To define the Fourier transform for functions on $SE(3)$, we have to use a complete orthogonal basis for functions on this group. The completeness of matrix elements (19) depends in part on the completeness of the rotation matrix elements $\tilde{U}^l_{mn}(A)$ on $SO(3)$ [16]. Using the unitary representations $U(g, p)$ (12) (for $s = 0, \pm 1, \pm 2 ...$), we can define the Fourier transform of functions on the motion group.

Definition. [6] *For any absolutely- and square-integrable complex-valued function $f(\mathbf{a}, A)$ on $SE(3)$ we define the Fourier transform as*

$$\mathcal{F}(f) \;=\; \hat{f}(p) \;=\; \int_{SE(3)} f(g) U(g^{-1}, p)\; d(g)$$

where $g = (\mathbf{a}, A) \in SE(3)$ and $d(g) = dA\, d^3\mathbf{a}$.

The matrix elements of the transform are given in terms of matrix elements (19) as

$$\hat{f}^s_{l',m';l,m}(p) \;=\; \int_{SE(3)} f(\mathbf{a}, A)\, \overline{U^s_{l,m;l',m'}(\mathbf{a}, A; p)}\, dA\, d^3\mathbf{a} \tag{21}$$

where we have used the unitarity property.

The inverse Fourier transform is defined by

$$f(g) \;=\; \mathcal{F}^{-1}(\hat{f}) \;=\; \frac{1}{2\pi^2} \int_{SE(3)} \mathrm{trace}(\hat{f}(p) U(g, p))\, p^2\, dp\ . \tag{22}$$

Explicitly

$$f(\mathbf{a}, A) \;=\; \frac{1}{2\pi^2} \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^{l} \int_0^\infty p^2\, dp\, \hat{f}^s_{l,m;l',m'}(p) U^s_{l',m';l,m}(\mathbf{a}, A; p) \quad . \tag{23}$$

We note that we may use any unitary equivalent representation $T^\dagger\, U(g, p)\, T$ (where T is a unitary transformation, which does not depend on g) to define the Fourier transform.

Convolution of functions. Recall that the convolution integral of functions $f_1, f_2 \in \mathcal{L}^2(SE(3))$ may be defined as

$$(f_1 * f_2)(g) \;=\; \int_{SE(3)} f_1(h)\, f_2(h^{-1} \circ g)\, d(h). \tag{24}$$

One of the most powerful properties of the Fourier transform of functions on $\mathbb{R}^N$ is that the Fourier transform of the convolution of two functions is the product of the Fourier transform of the functions. This property persists also for the convolution of functions on the group, namely

$$\mathcal{F}(f_1 * f_2) \;=\; \mathcal{F}(f_2)\,\mathcal{F}(f_1) \tag{25}$$

or, in the matrix form

$$(\mathcal{F}(f_1 * f_2))^s_{l',m';l,m}(p) \;=\; \sum_{j=|s|}^{\infty} \sum_{k=-j}^{j} (\hat{f}_2)^s_{l',m';j,k}(p)\,(\hat{f}_1)^s_{j,k;l,m}(p) \quad . \tag{26}$$

Parseval/Plancherel equality. This form of Parseval equality is valid

$$\int_{SE(3)} \mid f(\mathbf{a},A) \mid^2 \; dA d^3\mathbf{a} \;=$$

$$\frac{1}{2\pi^2} \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^{l} \int_0^{\infty} \mid \hat{f}^s_{l',m';l,m}(p) \mid^2 \; p^2\, dp \;=$$

$$= \frac{1}{2\pi^2} \int_0^{\infty} || \, \hat{f}(p) \, ||_2^2 \, p^2 \, dp \quad , \tag{27}$$

where the Hilbert-Schmidt norm of $\hat{f}(p)$ is given by

$$|| \, \hat{f}(p) \, ||_2^2 = \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^{l} \mid \hat{f}^s_{l',m';l,m}(p) \mid^2 \quad .$$

A similar relation holds for the inner product

$$\int_{SE(3)} (f(\mathbf{a},A), g(\mathbf{a},A))\, dA d^3\mathbf{a} \;=$$

$$\frac{1}{2\pi^2} \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^{l} \int_0^{\infty} \overline{\hat{f}^s_{l',m';l,m}(p)}\, \hat{g}^s_{l',m';l,m}(p)\, p^2 \, dp \;=$$

$$= \frac{1}{2\pi^2} \int_0^{\infty} \mathrm{trace}(\hat{f}^{\dagger}(p)\hat{g}(p)) p^2 \, dp \quad , \tag{28}$$

where $\hat{f}^{\dagger}_{l,m;l',m'} \;=\; \overline{\hat{f}_{l',m';l,m}}$ is the Hermitian conjugate.

6 An Overview of Applications

The Fourier transform for functions on $SE(3)$ is a computational and analytical tool that can be used in many application areas. These areas are described in detail in [6]. Here we give a brief overview.

In robotics, the author and coworkers have shown in a series of papers that the workspaces of manipulators can be generated using convolution on $SE(N)$ [7]. Previously the Fourier transform for $SE(2)$ was used in the inverse problem of manipulator design [13]. Analogous problems in the three-dimensional case can be solved given the definition of Fourier transform presented here [8].

The quantification of error propagation in serial linkages can also be described as a convolution on $SE(3)$ (see e.g. [6].

In polymer science, a quantity of interest is the probability density function (pdf) that describes the relative position and orientation of one end of a chain molecule with respect to the other. Partial differential equations that describe the evolution of this pdf on $SE(3)$ have been known for some time. In analogy with the way in which the classical Fourier transform is used to convert partial differential equations into a simpler form in Fourier space where they can be solved and converted back to the spatial domain, the Fourier transform for $SE(3)$ plays a similar role. See [9, 10].

In the study of liquid crystal mechanics, a large number of essentially rigid molecules change orientation in solution depending on an applied external field. The description of the positional and orientational density of molecules in the solution is a function on $SE(3)$, and the Fourier transform for $SE(3)$ serves as a tool for the decomposition of this function [6].

In image analysis (particularly template matching) the optimal correlation under rigid-body motions is sought. Each correlation may be calculated using the $SE(2)$ Fourier transform [14]. In three dimensions, a similar template matching problem is that of chemical binding in drug design. This may be an application of the $SE(3)$ Fourier transform in the future.

7 Conclusions

It is shown how the irreducible unitary representations of the group of rigid-body motions in three-dimensional space can be written in terms of the screw parameters of the motion. The Fourier transform of functions of motion (which is based on IURs) is a useful tool in applications. Most engineers are unfamiliar with this tool, and this paper provides an introduction to the topic.

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