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Parts Entropy and The Principal Kinematic Formula

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Abstract— **Assembly systems that are able to function in the presence of uncertainties in the positions and orientations of feed parts are, by definition, more robust than those that are not able to do so. Sanderson quantified this with the concept of "parts entropy," which is a statistical measure of the ensemble of all possible positions and orientations of a single part confined to move in a finite domain. In this paper the concept of parts entropy is extended to the case of multiple interacting parts. Various issues associated with computing the entropy of ensembles of configurations of parts with excluded-volume constraints are explored. The rapid computation of excluded-volume effects using the "Principal Kinematic Formula" from the field of Integral Geometry is illustrated as a way to potentially avoid the massive computations associated with brute-force calculation of parts entropy when many interacting parts are present.**

I. INTRODUCTION

In the field of assembly automation it has long been known that the design of machines that assemble parts should take advantage of part geometries [3], [4]. And robotic systems with minimal sensing can achieve goals if information about the parts and environment are known [6]. Systems that shake or otherwise randomize part positions and orientations (i.e., "poses") allow a collection of parts to sample the ensemble of all possible poses. Almost 25 years ago, Sanderson quantified the concept of part disorder by defining the concept of "parts entropy" [10]. In his analysis, he considered the tractable problem of noninteracting parts, thereby focusing on individual part entropies. In contrast, in this paper the concept of parts entropy is extended to the case of multiple interacting parts. Various issues associated with computing the entropy of ensembles of configurations of parts with excluded-volume constraints are explored. The rapid computation of excluded-volume effects using the "Principal Kinematic Formula" from the field of Integral Geometry is illustrated. References in the English language on this topic include [1], [12]

This is all relevant to assembly automation because assembly systems that are able to function in the presence of uncertainties in feed part positions and orientations are more robust than those that are not able to do so. And therefore having a way to compute the entropy of parts over a statistical ensemble of different feed configurations provides a measure of the capabilities of an assembly automation system. Such metrics quantify the relative effectiveness of such systems, and open up new possibilities for quantifiable design principles.

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The remainder of this paper is structured as follows. Section II reviews how the entropy of a single part is defined, and formulates the problem of computing the entropy of multiple parts. Section III adapts the principal kinematic formula from the field of Integral Geometry as a tool to efficiently compute part entropies for multiple parts. Section IV illustrates the method with an example of two planar disks and two ellipsoidal parts free to move in space.

II. PROBLEM FORMULATION

In the first subsection of this section, the concept of the entropy of a single part, as introduced by Sanderson, is reviewed and slightly modified in the context of different terminology. Given N parts that are sparsely scattered in an environment, the total parts entropy can be approximated as the sum of the individual part entropies. This approximation, while always an upper bound, is not accurate when the environment is more cluttered due to excluded volume (non-interpenetration) effects. These issues are addressed in the second subsection.

A. A Continuous Version of Sanderson's Parts Entropy

Information-theoretic entropy has been used by Sanderson to characterize parts for use in assembly operations [10]. The position and orientation of a part is described by a group element $g \in G$, where $G = SE(3)$ is the well-known group of rigid-body motions. If the part can attain poses with a particular frequency of occurrence at a particular time, which is described by a probability density function $f(g;t)$, then its entropy is defined as

$$S_f(t) = -\int_G f(g;t) \log f(g;t) dg \qquad (1)$$

where dg is the bi-invariant integration measure (i.e., volume element) with which to integrate (see e.g., [5]).

How is $f(g;t)$ defined ? Imagine that a single part is placed at random in an environment. And imagine that this is repeated over many trials. The result can be summarized with a probability density function $f(g;0)$, where 0 indexes the initial time. From each of the random initial conditions that are collectively defined as $f(g;0)$, an assembly task can be attempted. If successful, the assembly task will result in the part being placed at its desired location g_1. Therefore, over the ensemble of trials the probability will evolve from $f(g;0)$ to $\delta(g_1^{-1} \circ g)$ (a Dirac-delta function indicating that the part is placed in its desired location). The evolution of probability of part pose over this ensemble of trials can be described as $f(g;t)$, and the associated part entropy is $S_f(t)$. Clearly $S_f(t)$ decreases during a successful assembly

process. If $\delta(g_1^{-1} \circ g)$ is defined to allow for some small but finite part tolerance, then S_δ will be finite, and as $S_f(t) - S_\delta$ approaches zero, it means that the part is being successfully placed. $S_f(t)$ in (1) is essentially Shannon's entropy, and is a fundamental definition in information theory [9].

Sanderson adapted concepts from information theory to consider the entropy of rigid parts [10]. In this context, the probability density function, $f(g)$, describes the set of all possible poses of a rigid part ($g \in SE(3)$) as it is randomly placed on a substrate over an ensemble of experiments. The corresponding entropy is called "parts entropy". Sanderson measured parts entropy in bits of information assuming that sensors have a finite resolution in each generalized coordinate used to parameterize rigid-body motion. In contrast, continuous motion is addressed here.

A useful mathematical fact to keep in mind in the calculations that follow is that integration on unimodular groups such as $SE(3)$ is invariant to shifts and inversions:

$$\int_G f(g^{-1})dg = \int_G f(h \circ g)dg = \int_G f(g \circ h)dg = \int_G f(g)dg.$$

Here $h \in G$ is arbitrary.

B. Multiple Parts

An assembly process can be thought of one in which such that the initial probability densities for each of the i parts converge to their desired locations $\{g_i\}$:

$$f_i(g;t) \to \delta(g_i^{-1} \circ g) \quad \text{as} \quad t \to T$$

where T is the total time allowed for the assembly to be completed.

Given n parts, the i^{th} of which is free to be placed arbitrarily in an environment with frequency of occurrence given by $f_i(g_i;t)$, the entropy will be bounded by $S'(t) \leq \sum_{i=1}^{n} S_i(t)$ where S_i is the entropy of the i^{th} part computed independently (i.e., as if that were the only part). If, however, the environment is very cluttered and there is not a significant amount of free space, this bound will not be tight, and the entropy of the joint distribution of parts will have to be computed:

$$S'(t) = -\int_{G^n} f' \log f' dg_1 \cdots dg_n. \tag{2}$$

where

$$\int_{G^n} = \int_G \cdots \int_G,$$

and $f' = f'(g_1, g_2, ..., g_n; t)$.

Whereas in the independent case

$$f(g_1, g_2, ..., g_n; t) = \prod_{i=1}^{n} f_i(g_i; t) \tag{3}$$

where for each $i = 1, ..., n$

$$\int_G f_i(g_i;t)dg_i = 1$$

In general this simple form is not realistic and needs to be augmented to reflect the excluded volume of parts.

To begin, let's consider functions $d_i(\mathbf{x})$ that take the value of 1 on part i (when that part is centered on the identity reference frame) and zero otherwise. Therefore, if body i is moved by rigid body motion g_i, and likewise for body j, we can compute their overlap as

$$w_{ij}(g_i, g_j) = \int_{I\!R^3} d_i(g_i^{-1} \circ \mathbf{x}) d_j(g_j^{-1} \circ \mathbf{x}) d\mathbf{x}.$$

A general property of integration over all of three-dimensional space is that it is invariant under rigid-body motions. Therefore if, we make the change of variables $\mathbf{y} = g_i^{-1} \circ \mathbf{x}$, then we find that

$$w_{ij}(g_i, g_j) = w_{ij}(e, g_i^{-1} \circ g_j) = w_{ij}(g_j^{-1} \circ g_i, e).$$

Clearly when the two bodies do not overlap, $w_{ij} = 0$, and if they do overlap then $w_{ij} > 0$. This can be 'windowed' and made binary. If there is any overlap, let $W_{ij}(g_i^{-1} \circ g_j) = 1$, and if there is no overlap let $W_{ij}(g_i^{-1} \circ g_j) = 0$. Then the original $f(g_1, g_2, ..., g_n)$ in (3) could be replaced with one of the form

$$f'(g_1, g_2, ..., g_n; t) = C f(g_1, g_2, ..., g_n; t) \prod_{i<j}^{n} (1 - W_{ij}(g_i^{-1} \circ g_j)) \tag{4}$$

where C is the normalization required to make f' a probability density function, i.e., such that

$$\int_{G^n} f' dg_1 \cdots dg_n = 1.$$

Note that the product in (4) is not only over sequentially local pairs of bodies, but rather all bodies, where the '$i < j$' simply avoids double counting. In this way we have a tool for assessing the entropy of the unassembled state of parts in a confined environment. The change in entropy from the random ensemble of part conformations to the fully assembled product, $\Delta S = S_{f'} - S_\delta$, is a measure of how much disorder the assembly process reduces.

One way to assess the quality of the design of a product to be assembled is how much entropy must be overcome to assemble it from the disassembled ensemble to the final assembly state. In other words, a design for which ΔS is small is easy to assemble, and is therefore a good design. In contrast, one measure of how good an assembly automation system is is how large of a ΔS it can handle and still successfully assemble parts.

Naively, the computer-age way to compute probabilities such as f' and the associated entropy $S_{f'}$ would be to uniformly sample all possible positions and orientations of the moving body, and record the ratio of the number of intersects to the total. Computing power is sufficiently large these days that this computation could be done for two (or maybe three) planar bodies. But for three dimensional problems, where the space of motions for a rigid body is six dimensional, sampling each spatial dimension results in $O(N^6)$ motions. For each motion, intersections can be assessed by numerical integration (to compute a volume of overlap) or sampling in $O(N^3)$. Therefore, $O(N^9)$ operations would be used. This is an awemenous calculation. And if there are instead

m bodies, the result becomes $O(N^{6m+3})$. Fortunately, over the past century methods have been developed by a small group of pure mathematicians to compute integrals of interest analytically in closed form. In particular, a result called "The Principal Kinematic Formula" will be used here.

III. PRINCIPAL KINEMATIC FORMULAS

The field of Integral Geometry (also called Geometric Probability) is concerned with evaluating the probability that one body that moves uniformly at random will intersect with another body. Results from this field that are applicable to the computation of parts entropy are adapted here.

A. The Principal Kinematic Formula

Suppose that we have two convex bodies, H and K, viewed as subsets of $I\!R^n$. Recall that a body is convex if the line segment connecting two points in the body is contained in the body for all possible choices of pairs of points. Another way to say this is that if for any $\mathbf{x}, \mathbf{y} \in H$, then $t\mathbf{x} + (1-t)\mathbf{y} \in H$ for all $t \in [0,1]$.

The *indicator function* on any measurable body, B, (not necessarily convex and perhaps not even connected) is defined by:

$$i(B) = \begin{cases} 1 & \text{if } B \neq \emptyset \\ 0 & \text{for } B = \emptyset \end{cases}$$

Note that if g is some group of operations that acts on B without shrinking it to the empty set, then $i(g \cdot B) = i(B)$. The indicator function is one of many functions on a body that is invariant under rigid-body motion. Others include the volume of the body, the surface area (or perimeter in the two-dimensional case).

Let H be stationary, and let K be mobile. Let us denote $g \in G$ (where $G = SE(n)$ is the group of rigid-body motions for bodies contained in $I\!R^n$). Then, by definition,

$$gK = \{g \cdot \mathbf{x} | \mathbf{x} \in K\},$$

where if $g = (A, \mathbf{a})$ is the rigid-body motion with rotational part $A \in SO(n)$ and translational part $\mathbf{a} \in I\!R^n$, then the action of $G = SE(n)$ on $I\!R^n$ is $g \cdot \mathbf{x} = A\mathbf{x} + \mathbf{a}$.

It can be shown that the intersection of two convex bodies is a convex body [7]. Furthermore, the rigid-body motion (or even affine deformation) of a convex body does not change the fact that it is convex. Therefore, $H \cap gK$ will be a convex body, and $f_{H,K}(g) = i(H \cap gK)$ will be a compactly supported function on G that takes the value of 1 when H and the moved version of K (denoted as gK) intersect, and it will be zero otherwise. The function $f_{H,K}(g)$ has some interesting properties. Namely, if we shift the whole picture by an amount g_0, then this does not change the value of $f_{H,K}(g)$. In other words,

$$i(g_0(H \cap gK)) = i((g_0 H) \cap (g_0 gK)).$$

This means that if we choose $g_0 = g^{-1}$, then

$$f_{H,K}(g) = i((g^{-1}H) \cap K) = i(K \cap g^{-1}H) = f_{K,H}(g^{-1}).$$

In the special case when $H = K$ (i.e., they are two copies of the same body) then $f_{H,H}(g) = f_{H,H}(g^{-1})$, which is called a symmetric function.

More generally, "counting up" all values of g for which an intersection occurs is then equivalent to computing the integral

$$J = \int_G i(H \cap gK) dg. \tag{5}$$

A somewhat amazing result is that the integral J can be computed exactly using only elementary geometric properties of the bodies H and K without actually having to perform an integration over G. While the general theory has been developed by mathematicians for the case of bodies in $I\!R^n$ [11] and in manifolds on which some Lie group acts (see [12] and references therein), we are concerned only with the cases of bodies in $I\!R^2$ and $I\!R^3$. And what's more, integrals similar to (5) where the integrand is not $I(\cdot)$, but other so-called "mixed volumes" can also be computed in closed form [11], [12].

B. The Planar Case

A closed arc-length-parameterized curve of length L in the plane can be described (up to rigid-body motion) using the equation:

$$\mathbf{x}(s) = \begin{pmatrix} \int_0^s \cos\theta(\sigma) d\sigma \\ \\ \int_0^s \sin\theta(\sigma) d\sigma \end{pmatrix}$$

where

$$\theta(s) = \int_0^s \kappa(\sigma) d\sigma$$

is the counterclockwise-measured angle that the tangent to the curve makes with respect to the x-axis and $s \in [0, L]$.

The condition that the curve is closed is given by $\mathbf{x}(L) = \mathbf{0}$, and differentiability of the curve is guaranteed if $\theta(s)$ is continuous on $s \in [0, L]$ and $\theta(L) = 0$ (which ensures that the tangent at $\mathbf{x}(L)$ matches that at $\mathbf{x}(0)$. Continuity of the tangent direction can be relaxed to handle polygonal objects by allowing $\kappa(s)$ to be a sum of shifted Dirac delta functions, which makes $\theta(s)$ piecewise constant.

Regardless, for a convex body, the signed curvature, $\kappa(s)$, is always nonnegative.

For a simple, convex, closed curve, two global intrinsic quantities can be defined: the perimeter, L; the area enclosed by the curve, A. In this case the integral of normalized integral of total signed curvature,

$$\chi = \theta(L)/2\pi,$$

is equal to one. If this curve bounds a simply connected region, then $\chi = 1$ is both the indicator function and the Euler characteristic of that region. However, in more general cases in which a domain is not connected or not simply connected, then multiply closed curves define the boundaries of the domain and in these cases $\chi \neq 1$.

In the planar case, we can write (5) explicitly as

$$J = \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} i(H \cap g(a_1, a_2, \theta)K) da_1 da_2 d\theta \tag{6}$$

where the rotational part of $g = (R, \mathbf{a})$ is described by

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and the translational part is given by the vector $\mathbf{a} = [a_1, a_2]^T$. Together, these are described using homogenous transformation matrices of the form:

$$g(a_1, a_2, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & a_1 \\ \sin\theta & \cos\theta & a_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Such matrices form a group under the operation of matrix multiplication, and the bi-invariant integration measure on this group is defined using the following procedure. First, observe that the basis elements of the Lie algebra $\mathcal{G} = se(2)$ are:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that

$$g(a_1, a_2, \theta) = \exp(a_1 X_1)\exp(a_2 X_2)\exp(\theta X_3)$$

where $\exp(\cdot)$ is the matrix exponential. Furthermore, it can be shown by direct calculation that

$$g^{-1}\partial g = v_1^r X_1 + v_2^r X_2 + \omega X_3$$

and

$$(\partial g)g^{-1} = v_1^l X_1 + v_2^l X_2 + \omega X_3$$

where ∂g denotes any of the partials $\partial g/\partial a_i$ or $\partial g/\partial\theta$. This means that we can extract the relevant information from the above expressions into a vector as:

$$(g^{-1}\partial g)^\vee = \begin{pmatrix} v_1^r \\ v_2^r \\ \omega \end{pmatrix} \quad \text{or} \quad ((\partial g)g^{-1})^\vee = \begin{pmatrix} v_1^l \\ v_2^l \\ \omega \end{pmatrix}.$$

The bi-invariant volume element is then obtained as

$$dg = |\det J_r| da_1 da_2 d\theta = |\det J_l| da_1 da_2 d\theta$$

where

$$J_r(g) = \left[\left(g^{-1}\frac{\partial g}{\partial a_1}\right)^\vee, \left(g^{-1}\frac{\partial g}{\partial a_2}\right)^\vee, \left(g^{-1}\frac{\partial g}{\partial \theta}\right)^\vee\right]$$

and

$$J_l(g) = \left[\left(\frac{\partial g}{\partial a_1}g^{-1}\right)^\vee, \left(\frac{\partial g}{\partial a_2}g^{-1}\right)^\vee, \left(\frac{\partial g}{\partial \theta}g^{-1}\right)^\vee\right].$$

The fact that $J_r(g_0 g) = J_r(g)$ and $J_l(g g_0) = J_l(g)$ is obvious from their definition. The bi-invariance therefore follows follows from the fact that $|\det J_r| = |\det J_l|$, which in this particular case is equal to the number 1.

Theorem 1 (Blaschke, [2]): Given planar convex bodies H and K, then (6) evaluates as:

$$\int_{SE(2)} i(H \cap gK)dg = 2\pi[A(H) + A(K)] + L(H)L(K). \tag{7}$$

where $A(\cdot)$ is the area and $L(\cdot)$ is the perimeter of the body. Proof: See [2], [12].

In integral geometry, the statement of this theorem is in terms of the Euler characteristic rather than the indicator function. Recall that the Euler characteristic of a surface bounding a body B, which is denoted as $\chi(B)$, is a topological invariant of the surface. (In the planar case the Euler characteristic of a boundary curve is just the integral of signed curvature as defined earlier.) And in the case of convex bodies, the Euler characteristic and the indicator function are one in the same.

In the nonconvex case, we can bound the integral of interest from below and above by inscribing and circumscribing convex bodies inside and outside of H and K. Then computing (7) with the convex inscribed/circumscribed bodies will give lower and upper bounds on (7) for nonconvex H and K.

It is clear from (7) that the inscribed convex body should have as large of an area and perimeter as possible in order to obtain a lower bound that is as tight as possible. However, it is not clear what the trade-off between area and perimeter should be. Likewise, for the circumscribed convex body, the tightest upper bound will be obtained by a body of minimal area and perimeter.

C. The Spatial Case

It follows that if B has a continuous piecewise differentiable surface, ∂B, that we can compute

$$\int_{\partial B} dS = F(B)$$

(the total surface area). Furthermore, if κ denotes the Gaussian curvature at each point on the surface, we can compute (via the Gauss-Bonnet Theorem):

$$\int_{\partial B} \kappa dS = 2\pi\chi(B)$$

where $\chi(B)$ is the Euler characteristic. In the case of a convex spatial body, which necessarily is bounded by a surface of genus zero, $\chi(H) = 2 \cdot i(B)$.

In differential geometry a second kind of curvature is defined at every point on a surface. This is the *mean curvature*, m. The total mean sectional curvature is defined as

$$M(B) = \int_{\partial B} m dS.$$

In contrast to the indicator function, if we define

$$v_B(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \in B \\ 0 & \text{for } \mathbf{x} \notin B \end{cases}$$

then,

$$\int_{I\!R^n} v_B(\mathbf{x})d\mathbf{x} = \int_B d\mathbf{x} = V(B)$$

(the volume of B).

If spatial rigid-body motions are parameterized as

$$g(a_1, a_2, a_3; \alpha, \beta, \gamma) = \begin{pmatrix} A(\alpha, \beta, \gamma) & \mathbf{a} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

where $A(\alpha, \beta, \gamma)$ denotes the ZXZ Euler-angle parameterization, then the bi-invariant integration measure is, to within an arbitrary scaling constant, of the form [5]

$$dg = \sin\beta d\alpha d\beta d\gamma da_1 da_2 da_3.$$

It is derived in a similar way as for planar motions.

Theorem 2 (Blaschke, [2]): Given convex bodies H and K in $I\!R^3$, then

$$\begin{aligned} \int_{SE(3)} i(H \cap gK)dg &= 8\pi^2[V(H) + V(K)] \\ &+ 2\pi[A(H)M(K) + A(K)M(K)] \end{aligned} \quad (8)$$

where $A(\cdot)$ and $M(\cdot)$ are respectively the area and integral of mean curvature of the surface enclosing the body, and $V(\cdot)$ is the volume of the body. Proof: See [2], [12].

Again, we would really like to be able to compute (8) for nonconvex bodies, but it does not apply in that case, though integrals of the Euler characteristic can be obtained in that case.

IV. EXAMPLES

In this section the application of the principal kinematic formula to computing the entropy of parts in a cluttered environment consisting of two parts is illustrated. One part is taken to be fixed at the origin of a coordinate system, and another part is placed uniformly at random with its center of mass constrained to be within a sphere of radius R from the origin of the first part. This means that in the absence of the first part, the second has a volume of possible motions in $SE(n)$ given by

$$V = Vol(B^n(R)) \cdot Vol(SO(n))$$

where $Vol(B^n(R))$ is the volume of the ball defined by the interior of a sphere of radius R in n-dimensional space (which is πR^2 in $I\!R^2$ and $4\pi R^3/3$ in $I\!R^3$) and $Vol(SO(n))$ is the volume of the rotation group in n-dimensional space (which is 2π for $SO(2)$ and $8\pi^2$ for $SO(3)$) [5].

Therefore the the positional and orientational distribution of part # 2 computed in the absence of part # 1 would be

$$f(g) = \frac{1}{V}$$

for $g = (A, \mathbf{a}) \in SE(n)$ with $\|\mathbf{a}\| < R$, and $f(g) = 0$ otherwise.

The entropy of a single isolated part under these conditions is then

$$S_f = \log V.$$

In contrast, the total volume in $SE(n)$ that is available for part #2 to move if part # 1 is fixed in the environment, thereby limiting the range of possible motions of part #2, will be

$$V' = V - \int_{SE(n)} i(H \cap gK)dg$$

as long as R is larger than half of the sum of the maximal dimensions of the two parts. Otherwise, the effects of part #1 on limiting the motion may be even greater. With that caveat,

$$S_{f'} = \log V'. \quad (9)$$

Therefore, in this case we can completely avoid the computational complexity associated with computing (2) and (4) by using the principal kinematic formula from integral geometry.

A. Example 1: The Planar Case: Circular Disks in Planar Motion

Let part # 1 be a circular disk of radius r_1 fixed at the origin, and let part # 2 be a circular disk of radius r_2. If part # 2 were completely free to rotate, and free to translate such that its center stays anywhere in the large circle defined by radius R, then the part entropy would be

$$S = \log(2\pi^2 R^2).$$

In contrast, if all conditions are the same except that the constraint of no interpenetration is imposed, then

$$S' = \log(2\pi^2[R^2 - (r_1 + r_2)^2]),$$

which just removes the disallowed translations defined by the distance of the center of part # 2 from the origin in the range $[0, r_1 + r_2]$. This is a simple example that does not require any numerical computation of integrals of motion, or even the evaluation of the principal kinematic formula. But it serves to verify the methodology, since in this case

$$\begin{gathered} 2\pi[A(H) + A(K)] + L(H)L(K) = \\ 2\pi[\pi r_1^2 + \pi r_2^2] + (2\pi r_1)(2\pi r_2) = \\ 2\pi^2(r_1 + r_2)^2, \end{gathered}$$

which means that the adjustment to the computation of parts entropy from the principal kinematic formula (7) will be exactly the same as expected.

B. Example 2: Ellipsoids of Revolution in Spatial Motion

Consider an ellipsoid of revolution with dimensions of length a, a and b. The volume can be computed as:

$$V = \frac{4}{3}\pi a^2 b.$$

The values of surface area, F, and mean sectional curvature, M, for prolate and oblate ellipsoids have been reported in [8], along with a variety of other solids of revolution. In particular, if $a = R$ and $b = \lambda r$ with $0 < \lambda < 1$, then

$$F = 2\pi r^2 \left[1 + \frac{\lambda^2}{\sqrt{1-\lambda^2}} \log\left(\frac{1+\sqrt{1-\lambda^2}}{\lambda}\right)\right]$$

and

$$M = 2\pi r \left[\lambda + \frac{\arccos\lambda}{\sqrt{1-\lambda^2}} \right].$$

In contrast, when $\lambda > 1$,

$$F = 2\pi r^2 \left[1 + \frac{\lambda^2 \arccos(1/\lambda)}{\sqrt{\lambda^2 - 1}} \right]$$

and

$$M = 2\pi r \left[\lambda + \frac{\log(\lambda + \sqrt{\lambda^2 - 1})}{\sqrt{\lambda^2 - 1}} \right].$$

In the case of a sphere ($\lambda = 1$),

$$V = \frac{4}{3}\pi r^3; \quad F = 4\pi r^2; \quad M = 4\pi r.$$

This gives a concrete example of how to apply (8). And furthermore, given any two parts, one of which is fixed, the entropy of the freely moving part can be bounded from below and above by inscribing the largest possible ellipsoid of revolution in, and superscribing the smallest possible ellipsoid of revolution around, each part. Then complicated integrals such as (2) can then be avoided by using (8) and (9).

As a specific example to demonstrate this, consider the case of two spherical parts: part # 1 has radius r_1 and part # 2 has radius r_2. If part # 1 is fixed at the origin, and part #2 is free to move as long as its center does not go further than a distance R from the origin, then the volume of allowable motion of part #2 in $SE(3)$ will be

$$(8\pi^2)(4\pi/3)[R^3 - (r_1 + r_2)^3].$$

But (9) gives the amount of excluded volume in $SE(3)$ to be

$$8\pi^2[V(H) + V(K)] + 2\pi[A(H)M(K) + A(K)M(K)] =$$

$$8\pi^2[4\pi r_1^3/3 + 4\pi r_2^3/3] + 2\pi[(4\pi r_1^2)(4\pi r_2) + (4\pi r_2^2)(4\pi r_1)] =$$

$$(32\pi^3/3)(r_1^3 + r_2^3 + 3r_1^2 r_2 + 3r_1 r_2^2) =$$

$$(32\pi^3/3)(r_1 + r_2)^3.$$

And this too matches the direct analytical calculation for this simple example.

V. EXTENSIONS AND LIMITATIONS

The principal kinematic formula has been used to compute integrals of the form

$$J = \int_G i(H \cap gK) dg.$$

that arise when calculating the entropy of convex parts that can be placed uniformly at random. In integral geometry, generalized integrals of the form

$$J_1 = \int_G \mu(H \cap gK) dg$$

can be computed in closed form for bodies that are not convex, where μ can be the volume, Euler characteristic, surface area, mean curvature, or Gaussian curvature. This is not directly applicable to the current discussion, though it does open up intriguing possibilities.

A quantity that is not directly addressed in integral geometry is

$$J_2 = \int_G i(H \cap gK)\rho(g) dg$$

where $\rho(g)$ is a probability density function on G. This would be something that is useful for parts entropy calculations. The author is currently investigating this.

VI. CONCLUSIONS

The difficulty of an assembly task can be quantified using the concept of parts entropy. Sanderson's original formulation of this concept was for an individual isolated part. Issues that arise in the context of multiple parts are articulated in this paper. Methods of integral geometry are adapted in this paper and shown to be useful as a tool for computing the parts entropy of multiple parts. Open issues include how to adapt techniques from integral geometry to cases where the parts are not distributed uniformly at random, but rather have some prior probability densities. In addition, the issue of part entropies for articulated parts, rather than individual rigid parts, remains a challenging problem.

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