# Closed-Form Primitives for Generating Locally Volume Preserving Deformations

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*In this paper, methods for generating closed-form expressions for locally volume preserving deformations of general volumes in three dimensional space are introduced. These methods have applications to computer aided geometric design, the mechanics of materials, and realistic real-time simulation and animation of physical processes. In mechanics, volume preserving deformations are intimately related to the conservation of mass. The importance of this fact manifests itself in design, and in the realistic simulation of many physical systems. Whereas volume preservation is generally written as a constraint on equations of motion in continuum mechanics, this paper develops a set of physically meaningful basic deformations which are intrinsically volume preserving. By repeated application of these primitives, an infinite variety of deformations can be written in closed form.*

## 1 Introduction

This paper develops and enumerates deformations of objects which locally preserve volume. Given a three dimensional object described in Cartesian coordinates $\mathbf{x} = [x_1, x_2, x_3]^T$, a deformation maps these coordinates into a new set of coordinates: $\mathbf{X} = \mathbf{X}(\mathbf{x})$. A volume preserving deformation is one for which the volume of the object is kept constant after the deformation. This is written mathematically as:

$$\int_{\mathbf{x}} dx_1 dx_2 dx_3 = \int_{\mathbf{X}} dX_1 dX_2 dX_3 = \int_{\mathbf{x}} \det(\nabla_{\mathbf{x}} \mathbf{X}) dx_1 dx_2 dx_3. \quad (1)$$

where

$$\nabla_{\mathbf{x}} \mathbf{X} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{pmatrix}. \quad (2)$$

The quantity in Eq. (2) is called the deformation gradient.

A *locally* volume preserving deformation is one for which

$$\det(\nabla_{\mathbf{x}} \mathbf{X}) = 1. \quad (3)$$

A trivial example of a locally volume preserving deformation is a rigid body motion of the form $\mathbf{X}(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{r}$ where $\mathbf{Q}$ is a special orthogonal matrix.

A locally volume preserving deformation is also globally volume preserving, i.e., Eq. (1) is satisfied automatically by Eq. (3), but the converse is not generally true. Furthermore, compositions of locally volume preserving deformations are also locally volume preserving. That is, given two locally volume preserving deformations: $\mathbf{F}(\mathbf{x})$ and $\mathbf{G}(\mathbf{x})$, the compositions $\mathbf{F}(\mathbf{G}(\mathbf{x}))$ and $\mathbf{G}(\mathbf{F}(\mathbf{x}))$ are also locally volume preserving. This is a direct result of the chain rule, e.g.:

$$\nabla_{\mathbf{x}} \mathbf{F}(\mathbf{G}(\mathbf{x})) = \nabla_{\mathbf{y}} \mathbf{F}(\mathbf{y}) \nabla_{\mathbf{x}} \mathbf{G}(\mathbf{x}), \quad (4)$$

(where $\mathbf{y} = \mathbf{G}(\mathbf{x})$) and the fact that the determinant of a product of matrices is the product of the determinants, so

$$\det(\nabla_{\mathbf{x}} \mathbf{F}(\mathbf{G}(\mathbf{x}))) = \det(\nabla_{\mathbf{y}} \mathbf{F}(\mathbf{y})) \det(\nabla_{\mathbf{x}} \mathbf{G}(\mathbf{x})) = 1 \cdot 1 = 1. \quad (5)$$

Locally volume preserving deformations have significance in solid mechanics, biomechanics, and solid geometric modeling because of their intimate relationship to the conservation of mass of incompressible materials. For instance, in the analysis and simulation of the large deflections of many nonlinear elastic and plastic materials, Eq. (3) is incorporated as a constraint. Whether a solution is sought using analytical techniques or numerical techniques such as finite element methods, this constraint often arises. It is also the case that when one seeks to simulate the physical world in a realistic way, such constraints must be accounted for. Having a method of parametrizing classes of constant volume deformations could provide designers in mass-sensitive fields, such as aerospace engineering, a valuable tool for enumerating changes to current designs.

Many works in the computer graphics, mechanics, and geometric design literature have dealt with the deformation of solid models. Barr (1981) used angle-preserving deformations of superquadric surfaces to generate a wide variety of

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forms. Approximate methods for enforcing volume preservation where also examined. Barr (1984) extended these ideas to include general local and global deformations of arbitrary volumes. Sederberg and Parry (1986) developed free-form deformations of solid models based on trivariate Bernstein polynomials. Sederberg and Ferguson (1986) looked at the particular case of volume preserving deformations within this framework. It was found that volume preserving deformations based on cubic polynomials are limited to simple shear and scaling, and compositions thereof. Free-form deformations are also considered in Gudukbay and Ozguc (1990). Platt and Barr (1988) developed methods for representing the deformations of general solids based on continuum mechanics. In this approach, as is commonly done in solid mechanics, volume preservation is represented as a constraint which is imposed by using Lagrange multipliers (See Lai Rubin and Krempl, 1978; or Malvern, 1969).

Other works have considered the importance of volume preservation when simulating and analyzing biological systems. However, no general framework for describing volume preserving deformations has been developed. Miller (1988) simulates the locomotion of snakes and worms. This includes simultaneous longitudinal contraction and radial dilation so as to preserve volume. Chadwick et al. (1989) construct deformable animated figures which are made to look as real as possible (including volumetric constraints on contracting muscle). Arts et al. (1992) use locally volume preserving deformations with constant deformation gradients to model cardiac mechanics. Kier and Smith (1985) state the importance of local volume preservation in "muscular-hydrostats."

Whereas volume preservation is generally written as a constraint on equations of motion in continuum mechanics, this paper develops a set of physically meaningful basic deformations which are locally volume preserving. By repeated application of these primitives, an infinite variety of deformations can be written in closed form without the need for constraint equations. This opens up the possibility for more realistic real time simulation and animation of the physical world.

In Section 2, physically intuitive "Cartesian" deformations are defined and illustrated. In Section 3, two types of bending deformations based on planar offset curves are defined and used. Section 4 extends concepts from fluid mechanics, resulting in useful deformations. Section 5 shows examples of how combinations of closed form primitives can be used to generate more complicated locally volume preserving deformations.

## 2 Cartesian Deformations

In this section several locally volume preserving deformations which preserve parallelism between planar sections are examined. These are referred to here as Cartesian deformations since they are essentially motions of flat planar sections.

In Subsection 2.1, pure shear deformations are examined. In Subsection 2.2, pure "twist" deformations are examined. In Subsection 2.3, stretching and contraction are formulated.

**2.1 Simple Shear.** A simple shear deformation is one for which planar segments slide over each other without any rotation or change of their normal (denoted by the vector **n**). This deformation is expressed as:

$$\mathbf{S}(\mathbf{x}) = \mathbf{x} + d(\mathbf{x}\cdot\mathbf{n})\mathbf{t} \tag{6}$$

where **t** is any vector defined such that $\mathbf{n}\cdot\mathbf{t} = 0$, and $\mathbf{t}\cdot\mathbf{t} = 1$. If one imagines that $\mathbb{R}^3$ is composed of an infinite number of parallel planes, each with normal **n**, this deformation slides each of these planes a distance $d$ in the t direction. Since t lies in the plane with normal **n**, the effect is that each plane is translated within itself.

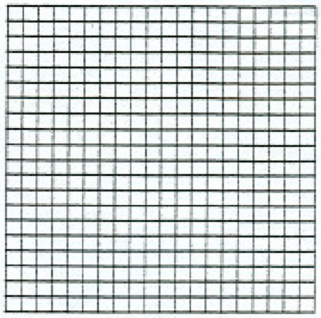

Fig. 1(*a*) A referential (undeformed) square

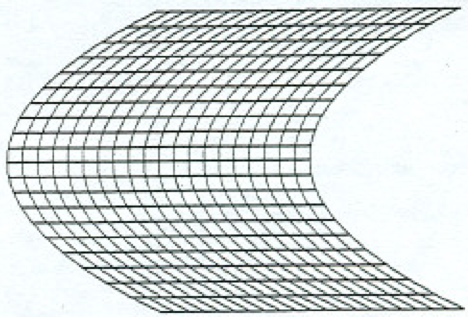

Fig. 1(*b*) A simple shear deformation: $\mathbf{S}_1(\mathbf{x})$

As an example, consider the shear deformation:

$$\mathbf{S}_1(\mathbf{x}) = \begin{pmatrix} x_1 + d(x_2) \\ x_2 \\ x_3 \end{pmatrix}. \tag{7}$$

This corresponds to Eq. (6) for the choice, $\mathbf{n} = \mathbf{e}_2$ and $\mathbf{t} = \mathbf{e}_1$. In fact, $\mathbf{S}_1(\mathbf{x})$ can be taken to be the standard form for a shear because Eq. (6) can be generated from an appropriate composition of Eq. (7) with rigid body motion.

By taking the partial derivatives $\partial\mathbf{S}/\partial x_i$ for $i = 1, 2, 3$, and computing the triple product of these vectors (which is the determinant of the deformation gradient), one finds it equals unity no matter what choice for $d$ is used. Figure 1 shows an example of a square before and after a shear deformation. In this case, $d(x_2) = x_2^2$.

As in all figures throughout this paper, the undeformed object is the square defined by the Cartesian product: $[-1, 1] \times [-1, 1]$.

A useful extension of simple shear is "fiber" shear. That is, instead of a whole plane translating, lines are translated independently of other parallel lines. An example is:

$$\mathbf{S}_2(\mathbf{x}) = \begin{pmatrix} x_1 + d(x_2, x_3) \\ x_2 \\ x_3 \end{pmatrix}. \tag{8}$$

An easy way to think of this deformation is that it is like the motion of uncooked spaghetti being removed from its box. Each strand can translate while remaining parallel to other strands.

**2.2 Twisting.** This deformation is similar to the simple shear in that three dimensional space is viewed as an infinite cascade of parallel planar sections. In this deformation, each plane is rotated about an axis which intersects the origin, and is parallel to the unit normal vector **n**. The plane is thus mapped back into itself. This is written as:

$$\mathbf{T}(\mathbf{x}) = (\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \mathrm{ROT}[\mathbf{n}\cdot\alpha(\mathbf{x}\cdot\mathbf{n})](\mathbf{x} - (\mathbf{x}\cdot\mathbf{n})\mathbf{n}). \tag{9}$$

The notation ROT[**n**, $\alpha$] represents the rotation matrix which rotates vectors about the unit vector **n** by an angle $\alpha$ in accordance with the right hand rule. In this case $\alpha$ is a function of the distance along the axis of rotation, so the resulting deformation is a twist. One can verify by direct calculation that this is also a locally volume preserving deformation.

Compositions of pure twist and shear along the same axis allows arbitrary translation and rotation of parallel planar segments.

**2.3 Elongation and Contraction.** This subsection again formulates deformations in which three dimensional space is viewed as an infinite cascade of parallel planes. Only now, each of these planes is translated along the normal direction instead of orthogonal to it. Furthermore, as material elements which occupy these planes are stretched or contracted in the normal direction, inverse operations must be performed in each of the planes. That is, to locally conserve volume, an element which is stretched in one direction must contract in an orthogonal direction. A functional relationship which satisfies this criteria is given by:

$$\mathbf{E}(\mathbf{x}) = f(\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \frac{g(\mathbf{x}\cdot\mathbf{n})}{f'(\mathbf{x}\cdot\mathbf{n})}[\mathbf{t}\cdot(\mathbf{x}-(\mathbf{x}\cdot\mathbf{n})\mathbf{n})]\mathbf{t} + \frac{1}{g(\mathbf{x}\cdot\mathbf{n})}[(\mathbf{t}\times\mathbf{n})\cdot(\mathbf{x}-(\mathbf{x}\cdot\mathbf{n})\mathbf{n})](\mathbf{t}\times\mathbf{n}), \quad (10)$$

where $f(\cdot)$, $f'(\cdot)$, $g(\cdot) > 0$, and $\mathbf{t}\cdot\mathbf{n} = 0$, where $\mathbf{t}\cdot\mathbf{t} = 1$. Other than these restrictions, the differentiable functions $f(\cdot)$, $\mathbf{g}(\cdot)$, and the vector **t** are arbitrary. Note that a $'$ represents differentiation of a function of a single variable. As an example, if $\mathbf{t} = \mathbf{e}_1$ and $\mathbf{n} = \mathbf{e}_2$ then one gets:

$$\mathbf{E}_0(\mathbf{x}) = \begin{pmatrix} \dfrac{g(x_2)}{f'(x_2)}x_1 \\ f(x_2) \\ \dfrac{x_3}{g(x_2)} \end{pmatrix}.$$

Two special cases of Eq. (10) are when $g(\cdot) = \sqrt{f'}(\cdot)$, and $\mathbf{g}(\cdot) = 1$. In the first case, Eq. (10) reduces to:

$$\mathbf{E}_1(\mathbf{x}) = f(\mathbf{x}\cdot\mathbf{n})\mathbf{n} + \frac{\mathbf{x}-(\mathbf{x}\cdot\mathbf{n})\mathbf{n}}{\sqrt{f'}(\mathbf{x}\cdot\mathbf{n})}. \quad (11)$$

As an illustration of the second case, when $\mathbf{n} = \mathbf{e}_1$ and $\mathbf{t} = \mathbf{e}_2$, we get:

$$\mathbf{E}_2(\mathbf{x}) = \begin{pmatrix} f(x_1) \\ \dfrac{x_2}{f'(x_1)} \\ x_3 \end{pmatrix}. \quad (12)$$

An example of this is shown in Fig. 2, where $f(x_1) = (1/2)x_1^2 + (3/2)x_1$.

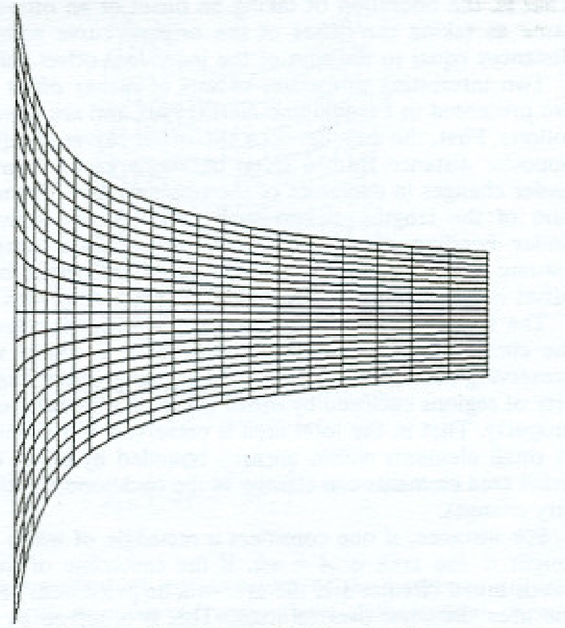

**Fig. 2 Nonuniform stretching: $\mathbf{E}_2(\mathbf{x})$**

## 3 Deformations Based on Offset Curves

This section develops a class of deformations based on the geometry of offset curves. Section 3.1 reviews basic properties of offset curves. Section 3.2 introduces pure bending deformations based on offset curves. Section 3.3 introduces the concept of offset shear-bending deformations.

**3.1 Properties of Planar Offset-Curves.** The offset of a planar "backbone" (or generator) curve is a curve which is parallel to the backbone. This is intimately related to the envelope of a circle whose center is moving along a backbone curve. Applications of offset curves include planning the trajectories of numerically controlled milling machines (Tiller and Hanson, 1984; Pham, 1992) and the locomotion of snake-like robots (Chirikjian and Burdick, 1991). Properties of offset curves have been studied in Farouki and Neff (1990). In this subsection, some properties of offset curves are reviewed. These properties will be used in the following subsections to define locally volume preserving "offset deformations."

In the plane, an offset curve, $\mathbf{o}(L)$, of a given backbone curve, $\mathbf{c}(L)$, is defined as:

$$\mathbf{o}(L) = \mathbf{c}(L) + r_0\mathbf{n}(L) \quad (13)$$

where $\mathbf{n}(L)$ is the unit normal to the curve $\mathbf{c}(L)$, and $r_0$ is a constant called the offset distance. For convenience, $L$ is taken to be the arclength of the curve $\mathbf{c}(\cdot)$.

The set of offset curves of a given curve can be thought of as curves which are all parallel to each other with different values of $r_0$. The notion that two curves are parallel is a reflective property. That is, if "A" is parallel to "B", then "B" is parallel to "A". To see that offset curves are in fact parallel to each other is straightforward.

Suppose we take the offset of an offset curve as follows:

$$\mathbf{i}(L) = \mathbf{o}(L) + r_1\mathbf{m}(L) \quad (14)$$

where $r_1$ is the offset distance of the second offset curve with respect to the first, and $\mathbf{m}(L)$ is the unit normal to $\mathbf{o}(L)$. By taking the derivative of (13) with respect to $L$ and using the Frenet-Serret equations (see Millman and Parker, 1977) for the planar case, one finds that

$$\frac{d\mathbf{o}}{dL} = (1 - r\kappa)\mathbf{u}. \quad (15)$$

where $\mathbf{u}(L)$ is the unit tangent vector to $\mathbf{c}(L)$ and $\kappa(L)$ is the curvature function of the curve. This means that the unit tangent to $\mathbf{o}(L)$, which is $(d\mathbf{o}/dL)/|d\mathbf{o}/dL|$, is the same as the unit tangent to **c**. It follows trivially that they then have the same unit normal. Thus, $\mathbf{m}(L) = \mathbf{n}(L)$, and so,

$$\mathbf{i}(L) = \mathbf{c}(L) + (r_0 + r_1)\mathbf{n}(L). \quad (16)$$

That is, the operation of taking an offset of an offset is the same as taking the offset of the original curve with offset distances equal to the sum of the individual offset distances.

Two interesting properties of sets of planar offset curves are presented in Farouki and Neff (1990), and are restated as follows. First, the area between two offset curves of equal but opposite distance from a given backbone curve is invariant under changes in curvature of the original curve. Second, the sum of the lengths of two such offset curves is invariant under bending of the backbone curve. These statements assume $1 - r\kappa > 0$. This assumption is important because offset curves develop cusps or self-intersect otherwise.

The first of the above mentioned properties is relevant to the current discussion in that our goal is to model volume preserving deformations. However, the area preserving property of regions enclosed by offset curves is a *global*, not *local* property. That is, the total area is preserved, but if one looks at small elements within an area bounded by offset curves, small area elements can change as the backbone curve geometry changes.

For instance, if one considers a rectangle of width $w$ and height $h$, the area is $A = wh$. If the centerline of the area bends into a circular arc, the area will be preserved, provided the area does not overlap itself. This is observed by taking the difference in area of two concentric circles with radii differing by $w$ and considering the portion of the resulting annular area which has a centerline of length equal to the original centerline. The area of the whole annulus is $A = \pi[(w + r)^2 - r^2]$, where $r$ is the radius of the inner circle. The centerline of the annular area is a third concentric circle with radius $r + w/2$ which bisects the annulus. Such a centerline will have circumference: $2\pi(r + w/2)$. The portion of the centerline of equal length as the original centerline is given by the ratio: $h/\pi(2r + w)$. The area of the segment of the annular area of centerline length $h$, is then

$$A = h/\pi(2r + w) \times \pi\left[(w + r)^2 - r^2\right] = hw.$$

However, the area elements on the inside of the circular arc will be compressed, while those on the outside will be stretched. This is clearly not locally area preserving. As stated earlier, the area preserving properties associated with planar offset curves is a global, not a local property. However, by extending the idea of an offset curve the following subsection develops two types of locally volume preserving deformations.

**3.2 Bending Deformations which Locally Preserve Volume.** This section develops a closed-form intrinsic parametrization of a class of bending deformations which are locally volume preserving. The following subsections develop two analytical models. These models are planar, though there are natural extensions to three dimensions (see Chirikjian, 1993). In Subsection 3.2.1 an analytical formulation based on "variable offset" curves and the area contained within these geometric structures is investigated. In Subsection 3.2.2, a deformation based on bending and reparametrization of a collection of constant offset curves is developed.

*3.2.1 Variable Offset Bending.* For a deformation to be locally area preserving, the area of each infinitesimal element must remain constant during the deformation. In order for the offset curve model to incorporate this feature, a generalized definition of a planar *variable offset bounded area* is defined below:

$$\mathbf{O}(\mathbf{x}) = \mathbf{c}(x_1) + r(x_1, x_2)\mathbf{n}(x_1). \quad (17)$$

This expression has a dual meaning. First, it can be considered as a deformation of a region in $x_1 - x_2$ space. Second, it is of the form of a set of offset curves with variable offset distance. Note that the parameter $x_1$ is not only a coordinate in the reference configuration, but also the arclength of the backbone curve. The above definition gives the function $r(x_1, x_2)$ two spatial degrees of freedom. The first is so that the offset can vary with the backbone curve parameter. The second is so that any point within the area bounded by two variable offset curves can be specified with the coordinates $(x_1, x_2)$. If for instance, the initial backbone is a straight line $\mathbf{c} = x_1\mathbf{e}_1$, and $r(x_1, x_2) = x_2$, then initially $(x_1, x_2)$ are the Cartesian coordinates for this slab, and they serve as referential coordinates for any continuously deformed configuration. See Lai Rubin Krempl (1978) or Malvern (1969) for definitions and details of referential descriptions of material deformation.

So that the model can incorporate the constraint of local volume (area) preservation no matter what kind of bending occurs, the function $r(x_1, x_2)$ is left undetermined for the time being. In order to enforce the local area preservation constraint, the following expression is observed:

$$\det(\nabla\mathbf{O}) = \left|\frac{\partial\mathbf{O}}{\partial x_1} \times \frac{\partial\mathbf{O}}{\partial x_2}\right| = 1. \quad (18)$$

That is, the area of each infinitesimal element defined by the Cartesian product: $[x_1, x_1 + dx_1] \times [x_2, x_2 + dx_2]$ must be independent of bending.

Evaluation of Eq. (18) by substituting in (17), and observing:

$$\frac{\partial\mathbf{O}}{\partial x_1} = (1 - r\kappa)\mathbf{u} + \frac{\partial r}{\partial x_1}\mathbf{n}; \quad \frac{\partial\mathbf{O}}{\partial x_2} = \frac{\partial r}{\partial x_2}\mathbf{n}, \quad (19)$$

and using the fact that $\mathbf{u}\cdot\mathbf{u} = \mathbf{n}\cdot\mathbf{n} = 1$ and $\mathbf{u}\cdot\mathbf{n} = 0$, yields

$$(1 - r\kappa)\frac{\partial r}{\partial x_2} = 1. \quad (20)$$

This expression is integrated with respect to $x_2$ to yield:

$$\left(r + \frac{1}{2}\kappa r^2\right) = x_2 + c(x_1). \quad (21)$$

The arbitrary function $c(x_1)$ is taken to be zero. Note that a nonzero choice of $c(x_1)$ corresponds to composing a shear deformation, i.e., replacing $x_2$ with $x_2 + c(x_1)$ is a shear in the same way that Eqs. (6) and (7) are.

Using the quadratic formula to solve for $r(x_1, x_2)$, one finds that:

$$r(x_1, x_2) = \frac{1 \pm (1 - 2\kappa(x_1)x_2)^{1/2}}{\kappa(x_1)}, \quad (22)$$

of which the negative root is used. This is used because as $\kappa(x_1)$ goes to zero, $r(x_1, x_2)$ should converge to $x_2$, i.e., the unbent configuration should correspond to the slab parametrized with Cartesian coordinates $(x_1, x_2)$. Note also that curvature of the backbone curve is always limited so that $1/2\kappa > x_2$ to avoid singularities.

As an example, consider the planar arclength parametrized backbone curve:

$$\mathbf{c}(x_1) = \left(\frac{1}{a}\sin ax_1, \frac{1}{a}(1 - \cos ax_1)\right)^T. \quad (23)$$

The deformed region is initially the square $[-1, 1] \times [-1, 1]$ shown in Fig. 1(*a*). In effect, Equations (17) and (22–23) define a deformation which bends the $x_1$ axis into a circular arc while preserving area locally. Figure 3 shows this for $a = \pi/12$.

*3.2.2 Offset Shear-Bending.* In this subsection, the properties of offset curves are exploited further. It is shown how

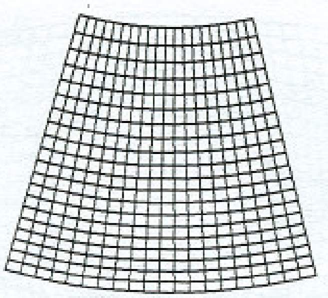

Fig. 3 Variable-offset bending: O(x)

allowing shear along directions parallel to the backbone curve preserves volume locally as the curve bends. This deformation is shown to be locally volume preserving when each of the collection of offset curves is parametrized by its own arc length instead of the backbone curve arclength, i.e., a reparametrization is required. Shear in this context is achieved by simply allowing translation of each offset curve, while maintaining parallelism with the backbone curve.

Given a set of offset curves where the backbone curve arclength is $s$, and offset distance is $x_2$:

$$\mathbf{O}(s, x_2) = \mathbf{c}(s) + x_2\mathbf{n}(s), \tag{24}$$

the variable $x_1$ can be defined to be the arc length of each offset curve:

$$x_1 = L(s, x_2) = \int_0^s \left|\frac{\partial \mathbf{O}}{\partial s}\right| ds = \int_0^s (1 - x_2\kappa(\sigma))d\sigma = s - x_2\theta(s), \tag{25}$$

where $\theta(s)$ is the integral of $\kappa(s)$. Solving Eq. (25) for $s$, one gets $s = \hat{s}(x_1, x_2)$. This expression is rarely algebraicly invertible. That is, in general the relationship $s = \hat{s}(x_1, x_2)$ cannot be found in closed form. Nevertheless, such a relationship (even if it is not expressible in closed form) does exist.

If we choose the coordinates, $x_1$ and $x_2$, and reparametrize the set of offset curves such that

$$\hat{O}(x_1, x_2) = \hat{O}(L(s, x_2), x_2) = O(s, x_2), \tag{26}$$

then $\hat{\mathbf{O}}(\mathbf{x})$ (viewed as a deformation) preserves local area independent of changes in curvature of the backbone curve. Proof of this fact is given below by direct calculation. The chain rule yields:

$$\frac{\partial \mathbf{O}}{\partial x_2} = \frac{\partial \hat{O}}{\partial x_1}\frac{\partial L}{\partial x_2} + \frac{\partial \hat{O}}{\partial x_2}; \quad \frac{\partial \mathbf{O}}{\partial s} = \frac{\partial \hat{O}}{\partial x_1}\frac{\partial L}{\partial s} \tag{27}$$

Writing Eqs. (27) in matrix form, and using the fact that $\partial L/\partial s = 1 - x_2\kappa(s)$ and $\partial L/\partial x_2 = -\theta(s)$, we get:

$$\begin{pmatrix} \dfrac{\partial \mathbf{O}}{\partial s} \\ \dfrac{\partial \mathbf{O}}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 - x_2\kappa & 0 \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} \dfrac{\partial \hat{O}}{\partial x_1} \\ \dfrac{\partial \hat{O}}{\partial x_2} \end{pmatrix}. \tag{28}$$

Inverting this equation, we find that:

$$\begin{pmatrix} \dfrac{\partial \hat{\mathbf{O}}}{\partial x_1} \\ \dfrac{\partial \hat{\mathbf{O}}}{\partial x_2} \end{pmatrix} = \frac{1}{1 - x_2\kappa} \begin{pmatrix} 1 & 0 \\ \theta & 1 - x_2\kappa \end{pmatrix} \begin{pmatrix} \dfrac{\partial \mathbf{O}}{\partial s} \\ \dfrac{\partial \mathbf{O}}{\partial x_2} \end{pmatrix}. \tag{29}$$

Since $x_3$ is unchanged by this deformation, one finds that:

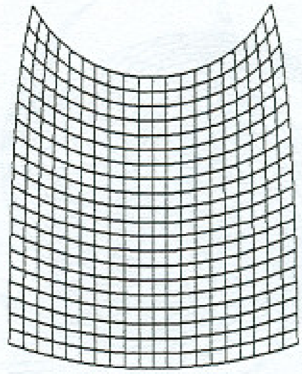

Fig. 4 Offset shear-bending: $\hat{\mathbf{O}}(\mathbf{x})$

$$\det(\nabla_x \hat{\mathbf{O}}) = \frac{1}{1 - x_2\kappa}\left(\frac{\partial \mathbf{O}}{\partial s} \times \frac{\partial \mathbf{O}}{\partial x_2}\right) \cdot \mathbf{e}_3 = (\mathbf{u} \times \mathbf{n}) \cdot \mathbf{e}_3 = 1. \tag{30}$$

Thus, independent of the curvature of the backbone curve, $\kappa$, local area is preserved using these deformations. Note however, that the restriction $1/\kappa > x_2$ must be made in order to avoid the singularities which occur when $1 = \kappa x_2$.

As an example, consider a backbone curve in Eq. (23). In this case, $\kappa(s) = a$, and (25) can be inverted to yield: $s = x_1/(1 - ax_2)$. Figure 4 shows this type of deformation applied to the same referential square as used earlier.

## 4 Deformations Based on Fluid Flow

Until now, this work has addressed purely geometric means of generating locally volume preserving deformations. In this section, elementary concepts from potential flow theory in fluid mechanics are used to define other types of locally volume preserving deformations. These deformations will be of the form $\mathbf{X}(\mathbf{x}, t)$. The time, $t$, is used to parametrize an evolving deformation. It is critical in deformations which mimic mechanics to use time. Once a solution is found, time can be frozen as needed to define particular deformations.

The general conditions for incompressible irrotational flow are respectively that the divergence and curl of the velocity field with respect to spatial coordinates, $\mathbf{X}$, are zero. That is,

$$\nabla_{\mathbf{X}} \cdot \mathbf{v} = 0 \quad \text{and} \quad \nabla_{\mathbf{X}} \times \mathbf{v} = \mathbf{0}. \tag{31}$$

By introducing a potential function, $\phi(\mathbf{X})$, such that the velocity field is given by: $\mathbf{v}(\mathbf{X}) = \nabla_{\mathbf{X}}\phi$, the incompressibility condition is equivalent to satisfying the Laplace Equation: $\nabla^2_{\mathbf{X}}\phi = 0$, and the irrotationality condition is automatically satisfied.

In planar steady incompressible potential flow, a stream function of the form $\psi(X_1, X_2)$ is also defined such that the *Eulerian* velocity of the flow measured at each point ($\mathbf{v} = [v_1, v_2]^T$) is of the form:

$$v_1(X_1, X_2) = \frac{\partial \psi}{\partial X_2} \quad \text{and} \quad v_2(X_1, X_2) = -\frac{\partial \psi}{\partial X_1}. \tag{32}$$

By introducing the stream function, the irrotationality condition is equivalent to solving the Laplace equation:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial X_1^2} + \frac{\partial^2\psi}{\partial X_2^2} = 0. \tag{33}$$

A large class of closed form stream functions and potentials exist which satisfy a wide variety of boundary conditions. However, one additional step must be taken in order to define a closed form deformation, $\mathbf{X}(\mathbf{x}, t)$. Namely, the referential (Lagrangian) and Eulerian velocities must be matched:

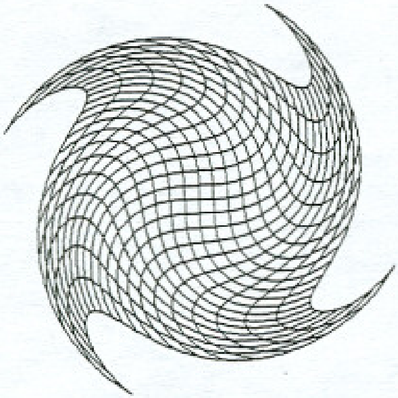

Fig. 5 A generalized vortex V(x)

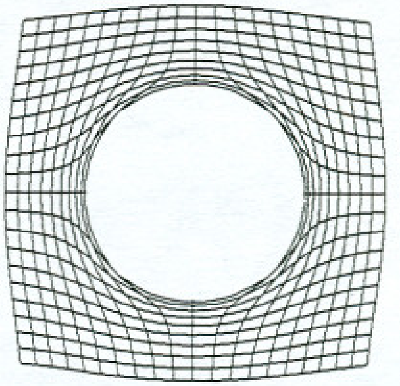

Fig. 6 A source $X_s(x)$

$$\frac{\partial \mathbf{X}}{\partial t} = \mathbf{v}(\mathbf{X}, t). \tag{34}$$

This must be true whether stream functions or potentials are used to define the velocity field.

This first order nonlinear partial differential equation must then be solved given appropriate initial conditions. In general, this cannot be done in closed form. However, the following subsections illustrate examples where it is possible. These subsections also illustrate how generalizations can be extracted from potential flow theory.

**4.1 Vortices.** The stream function of a vortex is written in terms of Cartesian coordinates as:

$$\psi = -\frac{K}{4\pi}\ln\left(X_1^2 + X_2^2\right). \tag{35}$$

Substituting this into (32) and (34), and using the initial conditions: $\mathbf{X}(\mathbf{x}, 0) = \mathbf{x}$, one finds that:

$$\mathbf{X}(\mathbf{x}, t) = \mathrm{ROT}\left[\mathbf{e}_3, c_0 + \frac{Kt}{2\pi\left(x_1^2 + x_2^2\right)}\right]\mathbf{x}. \tag{36}$$

The geometric meaning of this deformation is that circles of radius $r = (x_1^2 + x_2^2)^{1/2}$ from the center of the vortex rotate about the center by an angle $\alpha(r, t) = c_0 + (Kt/2\pi r^2)$. The constant $c_0$ induces rigid body rotation.

The function $\alpha(r, t)$ results from the irrotationality condition in fluid mechanics. However, if this condition is relaxed, Equation (36) generalizes to:

$$\mathbf{V}(\mathbf{x}) = \mathrm{ROT}\left[\mathbf{e}_3, \theta\left(\sqrt{x_1^2 + x_2^2}, x_3\right)\right]\mathbf{x}. \tag{37}$$

The function $\theta(\cdot)$ is an arbitrary differentiable function. The condition $\theta(0, 0) = 0$ is imposed without loss of generality because rigid body rotation can be composed.

Figure 5 shows this type of deformation applied to a rectangle. In this case, $\theta(r_1, 0) = r^2$.

**4.2 Sources and Sinks.** For a simple source/sink of strength $q$, the stream function is:

$$\psi = \frac{q}{2\pi}\tan^{-1}\left(X_2/X_1\right). \tag{38}$$

When $q > 0$, it is a source, when $q < 0$, it is a sink. Solving Eqs. (32) and (34) with the initial conditions: $\mathbf{X}(\mathbf{x}, 0) = \mathbf{x}$, one gets:

$$\mathbf{X}(\mathbf{x}, t) = \begin{pmatrix} \left(c(t) + x_1^2 + x_2^2\right)^{1/2} \dfrac{x_1}{\left(x_1^2 + x_2^2\right)^{1/2}} \\ \left(c(t) + x_1^2 + x_2^2\right)^{1/2} \dfrac{x_2}{\left(x_1^2 + x_2^2\right)^{1/2}} \end{pmatrix}, \tag{39}$$

where $c(t) = 4\pi t/q$. Figure 6 shows an example of this deformation for $c = 1$, and the deformation is denoted $\mathbf{X}_s(\mathbf{x})$.

This deformation is easily generalizable to the spatial case in three ways:

First, Equation (39) can be considered one slice of a "cylindrical" source where $c = c(x_3, t)$. In this way, tunnels can be made which fill a solid model by repeated application of the deformation composed with rigid body motions.

Second, one can imagine that if instead of a source, a radial stretch in the $x_1 - x_2$ plane is generated, the $x_3$ direction will have to be compressed. That is, if a point at radius $r = (x_1^2 + x_2^2)^{1/2}$ gets displaced to a new radius $r^* = h(r)$, then in order for a differential volume element written in cylindrical coordinates to have preserved volume: $r dr d\theta dz = r^* dr^* d\theta^* dz^*$. This means that if $\theta = \theta^*$ (just like a source) and $z^* = g(r, z)$, then

$$h(r)h'(r)\frac{\partial g}{\partial z} = r.$$

This can be solved to yield

$$g(r, z) = \frac{rz}{h(r)h'(r)} + C(r).$$

The $C(r)$ term can be viewed as a shear of concentric cylinders along their common axis. Since this is a special kind of fiber shear, we take $C(r) = 0$ knowing that such shears can be composed with the following more basic form (which is written in Cartesian coordinates):

$$\mathbf{P}(\mathbf{x}) = \begin{pmatrix} h\left(\left(x_1^2 + x_2^2\right)^{1/2}\right) \dfrac{x_1}{\left(x_1^2 + x_2^2\right)^{1/2}} \\ h\left(\left(x_1^2 + x_2^2\right)^{1/2}\right) \dfrac{x_2}{\left(x_1^2 + x_2^2\right)^{1/2}} \\ \dfrac{\left(x_1^2 + x_2^2\right)^{1/2} x_3}{h\left(\left(x_1^2 + x_2^2\right)^{1/2}\right) h'\left(\left(x_1^2 + x_2^2\right)^{1/2}\right)} \end{pmatrix}, \tag{40}$$

This type of deformation is inspired by the cylindrical nature of the planar source, but is really a type of stretching where planes normal to the stretch direction do not remain planes. The only restrictions on $h(\cdot)$ are that it be differentiable and $0 < h'(\cdot) < \infty$. We also impose the condition $h(0) = 0$ without loss of generality. If evaluating at $0 < \epsilon \ll 1$, one can avoid numerical division by zero by observing that $\epsilon/h(\epsilon) \approx 1/h'(\epsilon)$.

The third way to extend the concept of a source to the spatial case is to define a spherical potential of the form $\phi = [-m/4\pi(x_1^2 + x_2^2 - x_3^2)]$. This results in a deformation of the form:

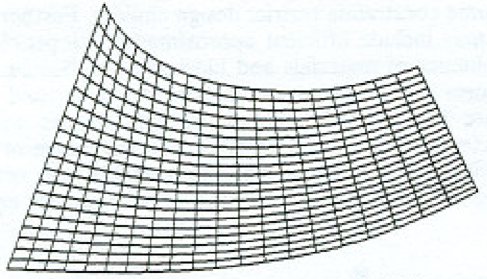

Fig. 7 Stretch and bending: $\mathbf{O}(\mathbf{E}_2(\mathbf{x}))$

$$\mathbf{X}(\mathbf{x}, t) = \begin{pmatrix} \left(\dfrac{3mt}{4\pi} + (x_1^2 + x_2^2 + x_3^2)^{3/2}\right)^{1/3} \dfrac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} \\ \left(\dfrac{3mt}{4\pi} + (x_1^2 + x_2^2 + x_3^2)^{3/2}\right)^{1/3} \dfrac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} \\ \left(\dfrac{3mt}{4\pi} + (x_1^2 + x_2^2 + x_3^2)^{3/2}\right)^{1/3} \dfrac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} \end{pmatrix}. \quad (41)$$

In both the planar and spatial cases, these source deformations provide a means by which voids can be formed in an otherwise simply connected solid model. In order to use them, a very small box, cylinder, or sphere must be introduced in the reference configuration to surround the singularity. In this way the topology of a solid model is changed, and the deformation makes large changes in geometry. This gives tremendous freedom to designers and solid modelers. That is, an initial simply connected blob of material can be made into a solid with as many voids as desired by multiple application of this deformation composed with rigid body motions.

## 5 Composition of Deformations

This section illustrates how composition of the volume preserving deformations presented earlier in this paper can be used to efficiently generate an infinite variety of volumetric shapes from a single referential volume.

A trivial example is the repeated composition of rigid body rotations to produce deformations of the form:

$$\mathbf{F}'(\mathbf{x}) = \mathbf{R}^T\mathbf{F}(\mathbf{R}\mathbf{x}).$$

If $\mathbf{R}^T = [\mathbf{t}, \mathbf{n}, \mathbf{t} \times \mathbf{n}]$ where $\mathbf{n}$ and $\mathbf{t}$ are unit vectors, then deformations represented in standard form can be used to generate more general deformations. For instance, looking at Eqs. (6) and (7), one finds that $\mathbf{S}(\mathbf{x}) = \mathbf{R}^T\mathbf{S}_1(\mathbf{R}\mathbf{x})$. Similarly, Eq. (10) can be generated from the standard elongation, $\mathbf{E}_0(\mathbf{x})$, by observing that $\mathbf{E}(\mathbf{x}) = \mathbf{R}^T\mathbf{E}_0(\mathbf{R}\mathbf{x})$.

In the following subsections, composition of deformations will be used to expand the variety of shapes that can be generated, and applications are presented.

**5.1 Stretch and Variable-Offset-Bending.** Consider a combined stretching and bending deformation. In this example, the stretch deformation in Eq. (12) is first applied, then the bending deformation defined by Eqs. (17) and (22–23) is applied. The resulting deformation is:

$$\mathbf{C}_1(\mathbf{x}) = \mathbf{O}(\mathbf{E}_2(\mathbf{x})). \quad (42)$$

The choice of primitive deformations resulting in Fig. 7 are:

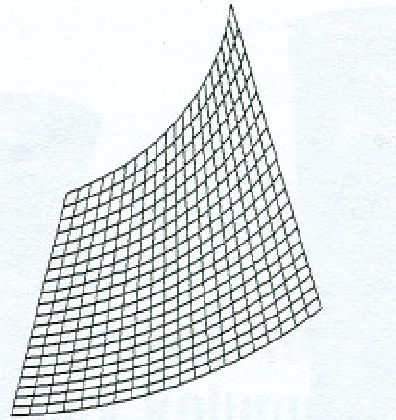

Fig. 8 Shear and variable-offset bending: $\mathbf{O}(\mathbf{S}_1(\mathbf{x}))$

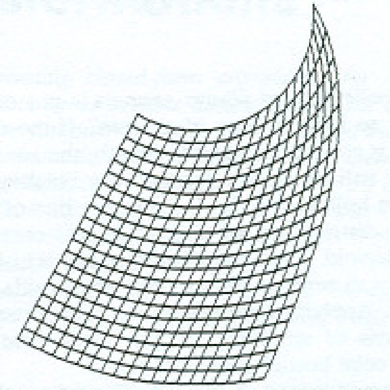

Fig. 9 Shear and offset shear-bending: $\hat{\mathbf{O}}(\mathbf{S}_1(\mathbf{x}))$

$$\mathbf{E}_2(\mathbf{x}) = \begin{pmatrix} \dfrac{1}{4}x_1^2 + \dfrac{3}{2}x_1 \\ \dfrac{x_2}{\dfrac{x_1}{2} + \dfrac{3}{2}} \\ x_3 \end{pmatrix} \quad (43)$$

and

$$\mathbf{O}(\mathbf{x}) = \begin{pmatrix} \dfrac{1}{a}\sin ax_1 - r(x_1, x_2)\sin ax_1 \\ \dfrac{1}{a}(1 - \cos ax_1) + r(x_1, x_2)\cos ax_1 \\ x_3 \end{pmatrix}, \quad (44)$$

where $r(x_1, x_2) = [1 - (1 - 2ax_2)^{1/2}/a]$, and $a = \pi/12$.

**5.2 Shear and Bending.** Figure 8 is a composition of shear and the bending deformation based on variable offsets defined in Subsection 3.2.1. This is written as $\mathbf{O}(\mathbf{S}_1(\mathbf{x}))$.

Figure 9 shows a composition of shear and the type of bending described in Subsubsection 3.2.2. That is, this deformation is of the form: $\hat{\mathbf{O}}(\mathbf{S}_1(\mathbf{x}))$ where $\mathbf{S}_1(\cdot)$ is defined in Equation (7), and $\hat{\mathbf{O}}(\mathbf{x})$ was defined in Eq. (33). In this figure, $d(x_2) = (1/2)x_2$.

**5.3 A Spatial Example.** As an application of the methods presented in this paper, consider the following scenario: An exotic looking bottle is needed for a new product, e.g., perfume, soft drink, etc. Aside from the cosmetic characteristics, the bottle will be molded from a specified amount of glass, and contain a specified volume of liquid. With the tools developed in this paper, an infinite variety of bottle designs can be created. Assume a cylinder is used as the referential

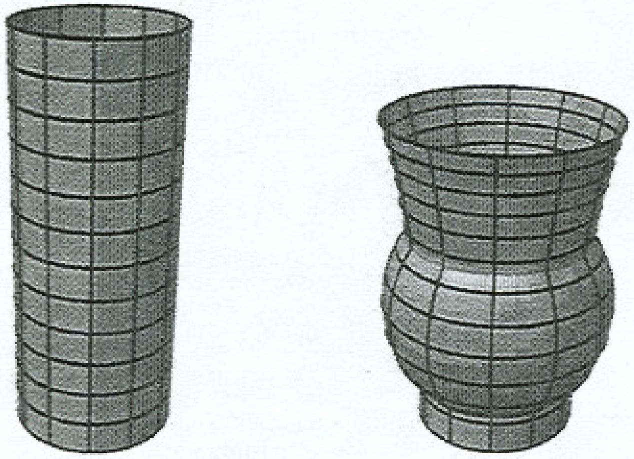

Fig. 10 Design of a bottle using multiple deformations

volume. A collection of planar sources is put on the axis of the cylinder to form a "line of sources". If the magnitude of the sources is constant along the length, the resulting hole in the cylinder will be cylindrical, and the resulting object will be a uniform hollow cylinder. Likewise, a line of sources with nonuniform distribution of magnitude will create a general axisymmetric void. The line of sources can terminate (source strength goes to zero) at one or both ends inside the referential cylinder. Applying nonuniform stretching and contraction along the axis of symmetry of this shape results in "hour glass" and "coke bottle" shaped volumes.

This is demonstrated in Fig. 10. The referential volume is a cylinder with radius 1/5, height 1, and base at the origin of the coordinate system. First a Cartesian stretch along the axis of symmetry is applied where $f(x) = \tan^{-1}(x)$. Then the object is translated down the axis of symmetry a distance of 0.3, a generalized planar source is applied with $c(x) = ((1/5)^2 - x^2)^{1/2}$ for $|x| < 1/5$ and $c(x) = 0$ otherwise. The object is then translated back up the axis of symmetry, and the object shown is the result.

While other methods, such as those in Celniker and Gossard (1989), or Cox et al. (1991) are applicable in this type of problem, the current formulation presents an alternative paradigm.

## 6 Conclusion

This paper has presented methods for generating and using locally volume preserving deformations which can be expressed in closed form. A combination of classical differential geometry and parametric geometry were used to generate these closed-form deformations. These deformations have applications in computer aided geometric design when mass and volume constraints restrict design choices. Further applications may include efficient approximations to problems in the mechanics of materials and biomechanics. Similarly, the closed-form primitives presented here could be used to approximate the physical universe in such diverse areas as robotics (e.g., modeling the kinematics and dynamics of robots with flexible actuators), and even virtual reality (e.g., real-time simulation of virtual contact with a very compliant environment).

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