Inverse Problems **12** (1996) 579–599. Printed in the UK

Fredholm integral equations on the Euclidean motion group

Gregory S Chirikjian

Department of Mechanical Engineering, Johns Hopkins University, Baltimore, MD 21218, USA

Received 21 March 1996

Abstract. In this work, methods for the solution of Fredholm equations of the first kind with convolution kernel are presented, where all the functions in the integral equation are functions on the Euclidean motion group, and the convolution product is defined relative to the group operation. An application in which such equations arise is examined in detail. The properties of the Fourier transform of scalar-valued functions on the Euclidean motion group are reviewed and applied to an exactly solvable example. Standard regularization techniques are then adapted and illustrated for cases in which exact solutions are not possible.

1. Introduction

Properties of the Fourier transform of scalar-valued functions on a given Lie group $(G, \circ)$ are used in this paper to solve equations of the form

$$\int_G k(g_\xi) f(g_\xi^{-1} \circ g_x)\, \mathrm{d}\mu(g_\xi) = (k * f)(g_x) = h(g_x) \tag{1}$$

where $g_x, g_\xi \in G$, $k(\cdot)$ and $h(\cdot)$ are given square-integrable scalar functions on G, $\mathrm{d}\mu(g_x)$ is a volume element at g_x, and $f(\cdot)$ is a function on G which is to be found either exactly or approximately.

This is a generalization of the Fredholm integral equation of the first kind with convolution kernel

$$\int_{-\infty}^{\infty} k(\xi) f(x-\xi)\, \mathrm{d}\xi = \int_{-\infty}^{\infty} k(x-\xi) f(\xi)\, \mathrm{d}\xi = (k * f)(x) = h(x) \tag{2}$$

except that G is a Lie group with operation $\circ$, and $\mathrm{d}\mu(\cdot)$ is a left-invariant volume element on G. This contrasts the specific case of $G = \mathbb{R}$ with the group operation $\circ = +$, and $\mathrm{d}\mu(g_\xi) = \mathrm{d}\xi$. For precise definitions of the terms used above see [10, 13, 22, 25, 31].

We will address the case when $G = SE(2)$—the two-dimensional special Euclidean group which describes all motions (translations and rotations) within the Euclidean plane. A subgroup of this group consists of translations, and the theory has as a subcase the standard theory of Fredholm integral equations of the first kind with convolution kernel.

The approach we will take is similar to the standard solution of Fredholm integral equations of the first kind with a convolution kernel like the one in equation (2). Namely, when possible a generalization of the Fourier transform will be used to convert this to a linear algebraic equation, which can either be solved exactly, or if the system is rank-deficient, is solved approximately using regularization methods.

0266-5611/96/050579+21$19.50

The reason for looking at the case when $G = SE(2)$ is that the mathematical problems formulated in this paper are motivated by applications in robot kinematics and motion planning, and group theoretical image processing and machine vision where the motion group plays a vital role. One such application is explored in detail in section 2, and others can be found in [1, 19].

We begin with a review of the Euclidean motion group of the plane, $SE(2)$, and provide motivation for exploring its unitary matrix representations, and the concept of the Fourier transform of functions on this group.

Any element $g \in SE(2)$ and its inverse g^{-1} can be represented as homogeneous transformation matrices of the form

$$g = \begin{pmatrix} R & \bar{b} \\ \bar{0}^T & 1 \end{pmatrix} \qquad g^{-1} = \begin{pmatrix} R^T & -R^T\bar{b} \\ \bar{0}^T & 1 \end{pmatrix} \tag{3}$$

where $\bar{b} \in \mathbb{R}^2$ and R is a 2×2 proper orthogonal matrix ($RR^T = 1$ and $\det(R) = +1$). The set of all matrices with the properties of R together with matrix multiplication is a group called the special orthogonal group, $SO(2)$. The group operation for $SE(2)$ is also matrix multiplication, and the product of two elements is defined by

$$g_1 \circ g_2 = \begin{pmatrix} R_1 & \bar{b}_1 \\ \bar{0}^T & 1 \end{pmatrix} \begin{pmatrix} R_2 & \bar{b}_2 \\ \bar{0}^T & 1 \end{pmatrix} = \begin{pmatrix} R_1R_2 & \bar{b}_1 + R_1\bar{b}_2 \\ \bar{0}^T & 1 \end{pmatrix}.$$

Each element of $SE(2)$ is parametrized in either rectangular or polar coordinates as

$$g(x_1, x_2, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & x_2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad g(r, \phi, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & r\cos\phi \\ \sin\theta & \cos\theta & r\sin\phi \\ 0 & 0 & 1 \end{pmatrix}.$$

The convolution product of square integrable functions on $SE(2)$ is expressed in rectangular coordinates in the form

$$\begin{aligned}(f_1 * f_2)(x_1, x_2, \theta) &= \int_{SE(2)} f_1(g(\xi_1, \xi_2, \alpha))\, f_2(g^{-1}(\xi_1, \xi_2, \alpha) \circ g(x_1, x_2, \theta))\, \mathrm{d}\mu(g(\xi_1, \xi_2, \alpha)) \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\xi_1, \xi_2, \alpha)\, f_2((x_1 - \xi_1)c\alpha + (x_2 - \xi_2)s\alpha \\ &\quad -(x_1 - \xi_1)s\alpha + (x_2 - \xi_2)c\alpha, \theta - \alpha)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\alpha\end{aligned}$$

where $c\alpha = \cos\alpha$ and $s\alpha = \sin\alpha$.

Using polar coordinates the convolution product has the form

$$\begin{aligned}(f_1 * f_2)(r, \phi, \theta) &= \int_{SE(2)} f_1(g(\rho, \gamma, \alpha))\, f_2(g^{-1}(\rho, \gamma, \alpha) \circ g(r, \phi, \theta))\, \mathrm{d}\mu(g(\rho, \gamma, \alpha)) \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} f_1(\rho, \gamma, \alpha) \\ &\quad \times f_2\left(\sqrt{r^2 + \rho^2 - 2r\rho c(\phi - \gamma)}, t(r, \rho, \phi, \alpha, \gamma), \theta - \alpha\right) \rho\, \mathrm{d}\rho\, \mathrm{d}\gamma\, \mathrm{d}\alpha\end{aligned}$$

where

$$t(r, \rho, \phi, \alpha, \gamma) = \mathrm{Atan2}(r\sin(\phi - \alpha) - \rho\sin(\gamma - \alpha), r\cos(\phi - \alpha) - \rho\cos(\gamma - \alpha))$$

and a normalized volume element/measure with which to integrate is defined by

$$\mathrm{d}\mu(g_\xi) = \frac{1}{(2\pi)^2} \rho\, \mathrm{d}\rho\, \mathrm{d}\gamma\, \mathrm{d}\alpha = \frac{1}{(2\pi)^2}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\alpha$$

where $g_\xi = g(\xi_1, \xi_2, \alpha) = g(\rho, \gamma, \alpha)$.

The function Atan2(y, x) is the two-argument inverse-tangent function which is equivalent to $\tan^{-1}(y/x)$ when x and y are positive, and generates a result in the range $-\pi$ to π depending on the signs of x and y.

In the above cases, notational abbreviations of the form $f(g(x_1, x_2, \theta)) = f(x_1, x_2, \theta)$ and $f(g(r, \phi, \theta)) = f(r, \phi, \theta)$ have been made since the meaning is clear from the context.

The fact that $\mathrm{d}\mu(g)$ as defined above for $SE(2)$ is a left and right invariant volume element (i.e. $\mathrm{d}\mu(g_\xi) = \mathrm{d}\mu(g_x \circ g_\xi) = \mathrm{d}\mu(g_\xi \circ g_x)$) gives the convolution product some interesting and useful properties†.

In particular, equation (1) can be written in the alternative form

$$\int_{SE(2)} k(g_x \circ g_\xi^{-1}) f(g_\xi)\, \mathrm{d}\mu(g_\xi) = (k * f)(g_x) = h(g_x) \tag{4}$$

by the change of variables $g_\xi \to g_x \circ g_\xi^{-1}$. This is true despite the fact that the group $SE(2)$ is not commutative and not compact.

One way to approach the problem of finding $f(g_x)$ is to treat it as one of a more general form:

$$\int_{SE(2)} K(g_x, g_\xi) f(g_\xi)\, \mathrm{d}\mu(g_\xi) = h(g_x). \tag{5}$$

This problem can be addressed in the usual ways that integral equations with non-self-adjoint non-separable multidimensional kernels are dealt with. For example, we can expand the unknown function in terms of a complete orthonormal series, and solve an associated set of linear algebraic equations. Using rectangular coordinates, equation (5) is rewritten as

$$\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x_1, x_2, \theta, \xi_1, \xi_2, \alpha) f(\xi_1, \xi_2, \alpha)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\alpha = (2\pi)^2 h(x_1, x_2, \theta)$$

and for the particular case of the convolution kernel in equation (4),

$$K(x_1, x_2, \theta, \xi_1, \xi_2, \alpha) = k(x_1 - \xi_1 c(\theta - \alpha) + \xi_2 s(\theta - \alpha),\\ x_2 - \xi_1 s(\theta - \alpha) + \xi_2 c(\theta - \alpha), \theta - \alpha).$$

Making the approximation

$$f(\xi_1, \xi_2, \alpha) \approx \exp[-(\xi_1^2 + \xi_2^2)/2] \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=-N}^{N} f_{ijk} H_i(\xi_1) H_j(\xi_2) \mathrm{e}^{\sqrt{-1}k\alpha}$$

where $H_i(x)$ is the ith normalized Hermite polynomial, the coefficients f_{ijk} can be found in the least squares sense in exactly the same way they would be for a standard integral equation. Multiplying both sides by $H_m(x_1)H_n(x_2)\mathrm{e}^{-(x_1^2+x_2^2)/2}\mathrm{e}^{\sqrt{-1}p\theta}$ and integrating over x_1,x_2, and θ means that the approximate solution of this convolution equation on $SE(2)$ using the Hermite–Fourier series approximation can be written as a system of linear equations of the form

$$\mathbf{K}\overline{f} = \overline{h} \qquad \text{or} \qquad \sum_{ijk} K_{mnpijk} f_{ijk} = h_{mnp}$$

which is solved to find f_{ijk}.

There are two problems with this approach. First, the matrix $\mathbf{K}$ has $[(2N+1)(N+1)^2]^2 = \mathrm{O}(N^6)$ elements, each of which is of the form

$$K_{mnpijk} = \int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x_1, x_2, \theta, \xi_1, \xi_2, \alpha) H_m(x_1) H_n(x_2) H_i(\xi_1) H_j(\xi_2)\\ \times \mathrm{e}^{-(x_1^2+x_2^2)/2}\mathrm{e}^{-(\xi_1^2+\xi_2^2)/2}\mathrm{e}^{\sqrt{-1}p\theta}\mathrm{e}^{\sqrt{-1}k\alpha}\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\alpha\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}\theta.$$

† One can easily derive this volume element and demonstrate its invariance using methods in [9] or [31].

These integrals will generally not be solvable in closed form, and multidimensional numerical integration can be time consuming and error prone [3]. Second, inverting the $(2N+1)(N+1)^2 \times (2N+1)(N+1)^2$ matrix $\mathbf{K}$ to find the coefficients f_{ijk} will require $O((N^3)^3) = O(N^9)$ computations. While this can be done if $N \approx 10$, if we want a good solution, $N \approx 100$ may be required.

These problems are compounded if one extends this approach to the Euclidean motion group of three-dimensional space, $SE(3)$. This is a six-dimensional Lie group for which each of the $O(N^{12})$ elements in the coefficient matrix $\mathbf{K}$ would require 12-dimensional integrals, and then $\mathbf{K}$ would have to be inverted, requiring $O(N^{18})$ computations.

Alternatively, sampling techniques developed recently for functions on Lie groups (e.g. [5, 23]) could potentially be used to approximate the integral equation at a finite number of points. From there, finite difference approximation methods and modifications thereof (e.g. [14, 15, 20]) could be applied as they are in the case of integral equations on the real line. Another approach would be to use efficient transforms developed for discrete groups [4, 27]. However, it is not clear how these approaches can be adapted to noncompact groups such as the Euclidean group since the number of points in the discretization may need to be large in each dimension.

In order to avoid these problems, and address the case of convolution kernels in the most natural way, the Fourier transform on the Euclidean group will be used.

The remaining sections of this paper are organized as follows. Section 2 examines a scenario in which the inverse problem in equation (1) arises. Section 3 reviews unitary representations of $SE(2)$ and the definition of the Fourier transform on $SE(2)$. Section 4 illustrates some of the useful properties of this Fourier transform, including an analogue to the convolution theorem and Parseval's equality for functions on the Euclidean group. Much of this material is a combination and restatement of that found in [29] and [30]. Section 5 provides an example in which the solution to equation (1) is generated exactly. Section 6 extends standard regularization techniques to solve equation (1) (or equivalently equation (4)). Section 7 is the conclusion.

2. An application in robotics

In order to motivate the need for solving the inverse problem stated in equation (1), we examine a problem from the field of robotics in detail.

A robotic manipulator arm is a device that is used to position and orient objects in the plane or in three-dimensional space. A manipulator is generally constructed of rigid links and actuators, such as motors or hydraulic cylinders, which cause all motions of the arm. If the actuators have only a finite number of states, as is the case with stepper motors or pneumatic cylinders, then the arm has a finite number of configurations and only a finite number of frames† are reachable by the hand. This is illustrated in figure 1 for a manipulator composed of linear/translational actuators. This manipulator is capable of only reaching eight positions and orientations in the plane. Such a manipulator is called a binary manipulator [2]. The set of all reachable positions and orientations is called the *workspace*. Clearly, in the case of discrete actuation the workspace is a discrete subset of $SE(2)$, though it is not a discrete subgroup.

For discretely actuated manipulators the *density* of reachable frames in $SE(N)$ determines how accurately a random position and orientation can be reached. This density

† A frame in space is competely determined by the position of its origin and its orientation relative to another frame.

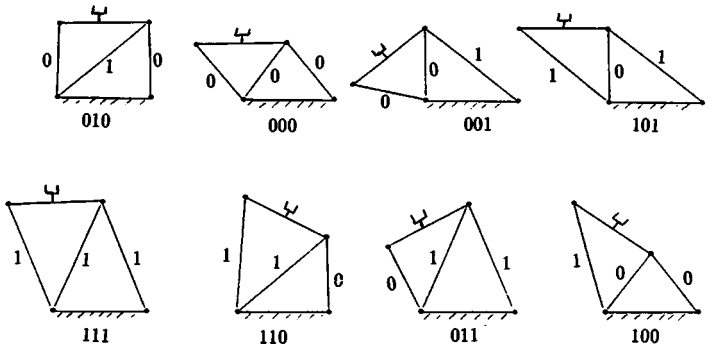

Figure 1. An eight-state binary manipulator.

information is also extremely important in planning the motions of discretely actuated manipulator arms [6]. Density is calculated directly by dividing a compact subset of $SE(N)$ containing the workspace into finite but small volume elements. The number of positions and orientations reachable by the end of the manipulator which lie in each volume element is stored. Dividing this number by the volume element size gives the average density in each element. An efficient method for the calculation of this density histogram is given in [7]. A smooth density function can be used to approximate the shape of this density histogram as in [8]. Note that the density function is always real-valued and positive.

It is an important aspect of the manipulator design problem to specify the density of reachable frames throughout the workspace. That is, areas which must be reached with great accuracy should have high density, and those areas of the workspace which are less important need less density. For relatively few actuators, this is achieved by enumerating reachable frames (positions and orientations) and using an iterative procedure as discussed in [2]. However, to compute this workspace density function using brute force and iterating is computationally intractable for large n. In fact, it requires K^n evaluations of the kinematic equations relating actuator state to the resulting end frame for a manipulator with n actuators each with K states. In figure 1, $K = 2$ and $n = 3$ so the problem is simple.

A grey scale of the density of frames reachable by a discretely actuated manipulator is shown in figure 2 with several configurations of the arm superimposed†. This manipulator is essentially a serial cascade of modules with the same kinematic structure as in figure 1, only now each leg has four states instead of two. Since each leg has four states (and thus the whole manipulator has $4^{30} \approx 10^{18}$ states) the workspace density cannot simply be computed using brute force. In fact, it would take years using current computer technology to enumerate all the positions and orientations of the frame attached to the end of the manipulator for each discrete configuration.

The concept of convolution of real-valued functions on $SE(N)$ provides a powerful computational tool for computing this density efficiently [7, 8]. If we imagine that the manipulator is divided into two connected parts, then a density function $\alpha(g)$ can be associated with those frames reachable by the end of the lower half of the manipulator, and a density function $\beta(g)$ can be associated with the end of the upper half of the manipulator. α is defined relative to the base frame, and β treats the frame at the end of the lower segment as the base frame. That is, $\alpha(g) = \beta(g)$ when the manipulator is cut into two equal parts

† A density function on $SE(2)$ can be written as $\rho(x, y, \theta)$. What is shown is really the integral of this over θ from $-\pi$ to π so that a planar picture results.

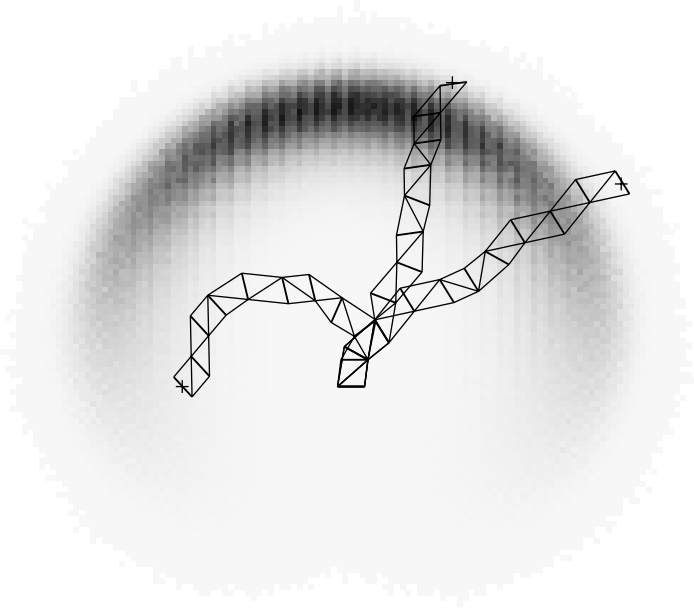

Figure 2. A discretely actuated manipulator with 4^{30} states.

and there are an even number of identical modules. However, α and β will not be the same function in more general scenarios. By adjusting kinematic parameters such that actuator strokes are limited or extended, the set of reachable frames (and thus the density) is altered. This is achieved mechanically by simply inserting or removing rigid stoppers that specify the physical actuator length corresponding to the discrete states.

While it may not be possible to calculate K^n frames to compute the density function of the workspace, it is often feasible to compute $K^{n/2}$ frames for each of the two segments. For the example discussed earlier, this would be of the order of a billion calculations, which can be done easily in less than an hour on a not-so-sophisticated computer. The density of the whole workspace is then generated by the convolution of these two functions:

$$\int_{SE(2)} \alpha(g)\beta(g^{-1} \circ h)\, \mathrm{d}\mu(g) = \gamma(h).$$

The geometry of why this is so follows below. Suppose there are two frames of reference—the base frame and one attached to the middle of the manipulator. Quantities described in the the base frame, $\mathcal{F}_1$, are denoted with a prime, while quantities described in the frame in the middle of the manipulator, $\mathcal{F}_2$, are denoted without a prime. Let the homogeneous transform matrix H describe the position and orientation of a third frame, $\mathcal{F}_3$, with respect to $\mathcal{F}_2$, and let $\mathcal{H}$ describe the position and orientation of $\mathcal{F}_2$ with respect to $\mathcal{F}_1$. Then the position and orientation of $\mathcal{F}_3$ with respect to $\mathcal{F}_1$ is

$$H' = \mathcal{H}H$$

as illustrated in figure 3.

If a real-valued function $\rho(H)$ is defined in the moving frame, then the same physical quantity can be represented in the fixed frame as $\rho'(H')$, where $\rho'(\cdot)$, is a different function from $\rho(\cdot)$. Knowing that the change of coordinates does not change the physical quantities

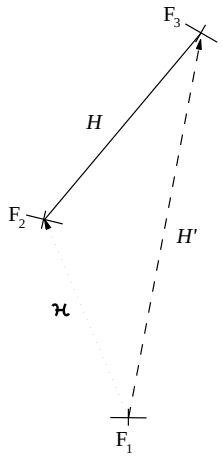

Figure 3. Concatenation of homogeneous transformations.

being described, one recognizes that

$$\rho'(H') = \rho(H).$$

Furthermore, because of the relationship between H and H' one finds that

$$\rho'(H') = \rho(\mathcal{H}^{-1}H').$$

In other words, in order to represent the function that is defined in the moving frame in the fixed frame coordinates, the homogeneous transform which describes the motion must be inverted and applied to fixed frame coordinates. Thus, the natural way to think of convolution of functions on the Euclidean group is that a function $\rho_1(\mathcal{H})$ is multiplied by $\rho_2'(H) = \rho_2(\mathcal{H}^{-1}H)$ and integrated over all values of $\mathcal{H}$:

$$(\rho_1 * \rho_2)(H) = \int_{SE(N)} \rho_1(\mathcal{H})\rho_2(\mathcal{H}^{-1}H)\,\mathrm{d}\mu(\mathcal{H}).$$

This calculation does not depend on the number of actuated modules in the manipulator, and allows us to compute the workspace density much more efficiently than by brute force. We can further subdivide the manipulator into smaller parts and perform multiple convolutions for even faster workspace density function generation.

Now, suppose that one of the halves of the manipulator has been designed ($\alpha(g)$ is specified), and the problem is to design the other half of the manipulator (find $\beta(g)$) so that the density function for the workspace of the whole manipulator comes as close as possible to a desired density function $\gamma(g)$. One must then solve the inverse problem

$$(\alpha * \beta)(g) = \gamma(g)$$

for $\beta(g)$. Once $\beta(g)$ is known, the methods developed in [2] can be used to find the appropriate kinematic parameters in the manipulator arm.

The remaining portion of this paper is devoted to the task of elegantly solving the above-stated inverse problem. But first, material from the pure mathematics literature is reviewed and then applied in later sections to the solution of this problem.

3. Unitary representations and the Fourier transform

Recall that one of the most powerful properties of the usual Fourier transform is that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the functions. This property extends to the concept of a Fourier transform for functions on $SE(2)$, but some review is first required. The reader familiar with the concept of the Fourier transform on $SE(N)$ can go directly to section 5.

Recall that the Fourier transform pair for a suitable function, $f(x)$, for $x \in \mathbb{R}$ is defined as

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) u(-x, \omega)\, \mathrm{d}x \qquad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) u(x, \omega)\, \mathrm{d}\omega \tag{6}$$

where $u(x, \omega) = \mathrm{e}^{\mathrm{i}\omega x}$. Note that $u(x + y, \omega) = \mathrm{e}^{\mathrm{i}\omega(x+y)} = \mathrm{e}^{\mathrm{i}\omega x}\mathrm{e}^{\mathrm{i}\omega y} = u(x, \omega)u(y, \omega)$. This is an example of a *homomorphism*. In general, a homomorphism is a mapping between two groups $h : (G, \circ) \to (H, \hat{\circ})$ such that $h(g_1 \circ g_2) = h(g_1)\hat{\circ}h(g_2)$. In particular, the function $u(\cdot, \omega)$ maps $(\mathbb{R}, +) \to (U, \cdot)$ for each $\omega \in \mathbb{R}$ where U is the set of complex numbers with unit modulus, and $\cdot$ is scalar multiplication.

The *convolution theorem* for functions on the real line states that: $\widehat{(f(x) * g(x))} = \hat{f}(\omega)\hat{g}(\omega)$. This is a direct result of the facts that $u(-(x + y), \omega) = u(-x, \omega)u(-y, \omega)$ and integration on the real line is translation invariant.

A *representation* of a group G is a homomorphism $T : G \to T(G) \subset GL(V)$. V is a vector space called the representation space, and $GL(V)$ is the group of all invertible linear transformations of V onto itself. $T(g)$ for $g \in G$ is expressed in a given basis of V as an invertible matrix, and

$$T(g_1 \circ g_2) = T(g_1)T(g_2) \qquad T(g^{-1}) = T^{-1}(g) \qquad T(e) = 1 \in GL(V).$$

Representations that can be expressed as unitary matrices ($U^{-1} = U^*$) in an orthonormal basis of V are called unitary representations. The function $u(x, \omega)$ is an example of a one-dimensional unitary representation. Henceforth in this paper the representation spaces used will be function spaces of the form $\mathcal{L}^n(P)$, which denotes the set of complex-valued functions $f(p)$ for which $\int_P |f(p)|^n\, \mathrm{d}\mu(p) < \infty$ where $p \in P$ for some manifold P. $\mathrm{d}\mu(p)$ is an appropriate volume element for each $p \in P$ (see [31] for a precise definition).

A unitary representation of $SE(2)$ (see [29, 30] for a general definition) is defined by the unitary operator

$$U_a(g)\tilde{f}(\overline{x}) = \mathrm{e}^{\mathrm{i}a(\overline{b}\cdot\overline{x})}\tilde{f}(R^T\overline{x}) \tag{7}$$

for each $g \in SE(2)$ of the form of equation (3). e^z is the scalar exponential function, $a \in \mathbb{R}^+$, $\mathrm{i} = \sqrt{-1}$, and $\overline{x} \cdot \overline{y} = x_1y_1 + x_2y_2$. The vector $\overline{x}$ is a unit vector ($\overline{x} \cdot \overline{x} = 1$), and $\tilde{f}(\cdot) \in \mathcal{L}^2(S^1)$ where S^1 is the unit circle. Since only one angle is required to parametrize a vector on the unit circle, $\overline{x} = (\cos\psi, \sin\psi)^T$, and $\tilde{f}(\overline{x}) = \tilde{f}(\cos\psi, \sin\psi) \equiv f(\psi)$. Henceforth we will not distinguish between $\tilde{f}$ and f.

By definition, group representations observe the homomorphism property, which in this case is seen as follows:

$$\begin{aligned} U_a(g_1)U_a(g_2)f(\overline{x}) &= U_a(g_1)(U_a(g_2)f(\overline{x})) = U_a(g_1)(\mathrm{e}^{\mathrm{i}a(\overline{b}_2\cdot\overline{x})}f(R_2^T\overline{x})) \\ &= \mathrm{e}^{\mathrm{i}a(\overline{b}_1\cdot\overline{x})}\mathrm{e}^{\mathrm{i}a(\overline{b}_2\cdot R_1^T\overline{x})}f(R_2^TR_1^T\overline{x}) \\ &= \mathrm{e}^{\mathrm{i}a(\overline{b}_1+R_1\overline{b}_2)\cdot\overline{x}}f((R_1R_2)^T\overline{x}) = U_a(g_1 \circ g_2)f(\overline{x}). \end{aligned}$$

Any function $f(\psi) \in \mathcal{L}^2(S^1)$ can be expressed as a weighted sum of orthonormal basis functions as $f(\psi) = \sum_n a_n e^{in\psi}$. Likewise, the matrix elements of the operator $U_a(g)$ are expressed in this basis as

$$u_{mn}(g, a) = (e^{im\psi}, U_a(g)e^{in\psi}) \qquad \forall\, m, n \in \mathbb{Z}$$

where the inner product $(\cdot, \cdot)$ is defined as

$$(f_1, f_2) = \frac{1}{2\pi}\int_0^{2\pi} f_1(\psi)\overline{f_2(\psi)}\,d\psi.$$

It is easy to see that $(U_a(g)f_1, U_a(g)f_2) = (f_1, f_2)$, and that $U_a(g)$ is therefore unitary with respect to this inner product.

The matrix with elements u_{mn} is 'infinite-dimensional'. Furthermore, the matrix of a unitary operator expressed in an orthonormal basis is a unitary matrix, which means $u_{nm}^{-1} = \overline{u_{mn}}$.

A number of works including [17, 24, 26, 30] have shown that the matrix elements of this representation are given by

$$u_{mn}(g(r, \phi, \theta), a) = i^{n-m}e^{-i[n\theta+(m-n)\phi]}J_{n-m}(ar) \tag{8}$$

where $J_\nu(x)$ is the νth order Bessel function.

From this expression, and the fact $U_a(g)$ is a unitary representation, we have

$$\begin{aligned} u_{mn}(g^{-1}(r, \phi, \theta), a) &= u_{mn}^{-1}(g(r, \phi, \theta), a) \\ &= \overline{u_{nm}}(g(r, \phi, \theta), a) \\ &= i^{n-m}e^{i[m\theta+(n-m)\phi]}J_{m-n}(ar). \end{aligned} \tag{9}$$

Henceforth no distinction will be made between the operator $U_a(g)$ and the corresponding infinite-dimensional matrix with elements $u_{mn}(g, a)$.

Unitary representations of higher-dimensional Euclidean groups, $SE(N)$, follow in a similar way, although the cases of the most practical interest are $SE(2)$ and $SE(3)$.

Given this background we are ready for the following definition.

Definition [29]. The Fourier transform of a rapidly decreasing function† $F \in \mathcal{L}^2(SE(2))$ and the inverse transform are defined as

$$\mathcal{F}(F) = \hat{F}(a) = \int_{SE(2)} F(g)U_a(g^{-1})\,d\mu(g)$$

and

$$\mathcal{F}^{-1}(\hat{F}) = F(g) = \int_0^\infty \mathrm{trace}(\hat{F}(a)U_a(g))a\,da.$$

Recall that the trace of a matrix $A \in \mathbb{R}^{N\times N}$ is the sum of the diagonal elements: $\mathrm{trace}(A) = \sum_{i=1}^N a_{ii}$, and the trace is invariant under transformations of the form QAQ^T where $Q \in SO(N)$. That is, $\mathrm{trace}(A) = \mathrm{trace}(QAQ^T)$. Furthermore $\mathrm{trace}(AB) = \mathrm{trace}(BA)$.

As with the Fourier transform of functions on $\mathbb{R}^N$,

$$\mathcal{F}\mathcal{F}^{-1}(\hat{F}) = \hat{F} \qquad \mathcal{F}^{-1}\mathcal{F}(F) = F$$

and so we write symbolically

$$\mathcal{F}\mathcal{F}^{-1} = \mathcal{F}^{-1}\mathcal{F} = 1$$

† $F(\cdot)$ is rapidly decreasing if $\lim_{r\to\infty} r^n F(g(r, \phi, \alpha)) = 0$ for all $n \in \mathbb{Z}^+$. Examples include functions $F(\cdot)$ which are zero outside of a compact subset of $SE(2)$ (i.e. if $F(\cdot)$ has compact support), and $F(g) = e^{-a^2r^2}f(\phi, \alpha)$ where $f(\cdot)$ is bounded.

where 1 is the identity operator. A proof that these identities hold is given in [29]. The fact that the inverse transform works depends on $U_a(g)$ being irreducible, and the fact that it is unitary allows us to write $U_a(g^{-1}) = U_a^*(g)$ instead of computing the inverse of an infinite-dimensional matrix. In subsection 4.2 we show that the inversion formula works.

The matrix elements of the transform can be calculated using the matrix elements of $U_a(g)$ defined in equation (8) as

$$\hat{F}_{mn}(a) = (\mathrm{e}^{\mathrm{i}m\psi}, \hat{F}(a)\mathrm{e}^{\mathrm{i}n\psi}) = \int_{SE(2)} F(g) u_{mn}(g^{-1}, a)\, \mathrm{d}\mu(g).$$

Likewise, the inverse transform can be written in terms of elements as

$$f(g) = \int_0^\infty \hat{F}_{mn}(a) u_{nm}(g, a) a\, \mathrm{d}a$$

where summation notation is used for $n, m \in [-\infty, \infty]$.

4. Properties of convolution and Fourier transforms of functions on $SE(2)$

In subsection 4.1 it is shown that the Fourier transform defined in section 3 possesses the convolution property in an analogous way to the usual Fourier transform. In subsection 4.2 the inversion formula is proven. In subsection 4.3 Parseval's inequality is proven. In subsection 4.4 some of the operational properties of the Fourier transform pair for functions on $SE(2)$ are exemplified.

4.1. The convolution theorem

Let us assume that there are real scalar-valued functions $\rho_1(\cdot)$, $\rho_2(\cdot) \in \mathcal{L}^2(SE(2))$. Recall that one of the most powerful properties of the Fourier transform of functions on $\mathbb{R}^N$ is that the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the functions. This property extends to the concept of a Fourier transform for functions on $SE(2)$. The proof of this fact presented here is due to Sugiura [29].

Given that

$$(\rho_1 * \rho_2)(g_x) = \int_{SE(2)} \rho_1(g_\xi)\rho_2(g_\xi^{-1} \circ g_x)\, \mathrm{d}\mu(g_\xi)$$

one gets

$$\begin{aligned}\mathcal{F}(\rho_1 * \rho_2) &= \int_{SE(2)} (\rho_1 * \rho_2)(g_x) U_a(g_x^{-1})\, \mathrm{d}\mu(g_x) \\ &= \int_{SE(2)} \left(\int_{SE(2)} \rho_1(g_\xi)\rho_2(g_\xi^{-1} \circ g_x)\, \mathrm{d}\mu(g_\xi) \right) U_a(g_x^{-1})\, \mathrm{d}\mu(g_x). \qquad (10)\end{aligned}$$

Switching the order of integration, one gets

$$\mathcal{F}(\rho_1 * \rho_2) = \int_{SE(2)} \left(\int_{SE(2)} \rho_2(g_\xi^{-1} \circ g_x) U_a(g_x^{-1})\, \mathrm{d}\mu(g_x) \right) \rho_1(g_\xi)\, \mathrm{d}\mu(g_\xi). \qquad (11)$$

Since $\mathrm{d}\mu(\cdot)$ is left and right invariant

$$\begin{aligned}\int_{SE(2)} F(g_\xi \circ g_x)\, \mathrm{d}\mu(g_x) &= \int_{SE(2)} F(g_x \circ g_\xi)\, \mathrm{d}\mu(g_x) \\ &= \int_{SE(2)} F(g_x^{-1})\, \mathrm{d}\mu(g_x) \\ &= \int_{SE(2)} F(g_x)\, \mathrm{d}\mu(g_x)\end{aligned}$$

for any function $F \in \mathcal{L}^2(SE(2))$. These facts allow us to write the inner integral in equation (11) as

$$\int_{SE(2)} \rho_2(g_\xi^{-1} \circ (g_\xi \circ g_x)) U_a((g_\xi \circ g_x)^{-1})\, \mathrm{d}\mu(g_x) = \int_{SE(2)} \rho_2(g_x) U_a(g_x^{-1} \circ g_\xi^{-1})\, \mathrm{d}\mu(g_x).$$

Since $U_a(G)$ is a representation of $SE(2)$,

$$U_a(g_x^{-1} \circ g_\xi^{-1}) = U_a(g_x^{-1}) U_a(g_\xi^{-1})$$

and so

$$\begin{aligned}\mathcal{F}(\rho_1 * \rho_2) &= \int_{SE(2)} \left(\int_{SE(2)} \rho_2(g_x) U_a(g_x^{-1}) U_a(g_\xi^{-1})\, \mathrm{d}\mu(g_x) \right) \rho_1(g_\xi)\, \mathrm{d}\mu(g_\xi) \\ &= \left(\int_{SE(2)} \rho_2(g_x) U_a(g_x^{-1})\, \mathrm{d}\mu(g_x) \right) \left(\int_{SE(2)} \rho_1(g_\xi) U_a(g_\xi^{-1})\, \mathrm{d}\mu(g_\xi) \right) \\ &= \mathcal{F}(\rho_2)\mathcal{F}(\rho_1) = \hat{\rho}_2(a)\hat{\rho}_1(a).\end{aligned}$$

The order in which $U_a(g_x^{-1})$ and $U_a(g_\xi^{-1})$ appear is important because they are representations of $SE(2)$, which is not a commutative group. As seen above, the fact that U_a is a representation is critical for the separation required for the convolution theorem to hold. The non-commutative nature of $SE(2)$ expressed in $U_a(g)$ is responsible for the reversed order of the product of the Fourier transforms. Some authors (e.g. [10, 11]) define the Fourier transform of functions on groups differently so that the order of the product of the transform is the same as the order of the convolved functions. We use the definition of Fourier transform given in section 3 because it is the most analogous to the standard Fourier transform.

4.2. Proof of the inversion formula

We now present a proof that the Fourier inversion formula for functions on $SE(2)$ actually works. This proof is coordinate dependent to avoid the introduction of additional mathematical machinery. Summation notation is again used, i.e. repeated indices indicate summation from $-\infty$ to ∞. A very elegant coordinate-independent proof can be found in [29]. The proof presented here is a variant on one found in [30].

Rapidly decreasing functions $f(g) \in \mathcal{L}^2(SE(2))$ where $g = g(r, \phi, \theta)$ can be expressed in a series of the form

$$f(g) = f(r, \phi, \theta) = F_{jk}(r) \mathrm{e}^{-\mathrm{i}j\phi} \mathrm{e}^{-\mathrm{i}k\theta}.$$

The matrix elements of the Fourier transform of this function (as defined in section 3) are

$$\begin{aligned}\hat{f}_{mn} &= \int_G f(g) u_{mn}(g^{-1}, a)\, \mathrm{d}\mu(g) \\ &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_0^{\infty} (F_{jk}(r) \mathrm{e}^{-\mathrm{i}j\phi} \mathrm{e}^{-\mathrm{i}k\theta}) (\mathrm{i}^{n-m} \mathrm{e}^{\mathrm{i}[m\theta + (n-m)\phi]} J_{m-n}(ar)) r\, \mathrm{d}r\, \mathrm{d}\phi\, \mathrm{d}\theta.\end{aligned}$$

Rearranging the integrals, this is rewritten as

$$\begin{aligned}&\left(\mathrm{i}^{n-m} \int_0^{\infty} F_{jk}(r) J_{m-n}(ar) r\, \mathrm{d}r \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}j\phi} \mathrm{e}^{\mathrm{i}(n-m)\phi}\, \mathrm{d}\phi \right) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{e}^{\mathrm{i}m\theta}\, \mathrm{d}\theta \right) \\ &\quad = \delta_{k,m} \delta_{j,n-m} \left(\mathrm{i}^{n-m} \int_0^{\infty} F_{jk}(r) J_{m-n}(ar) r\, \mathrm{d}r \right) \\ &\quad = \mathrm{i}^{n-m} \int_0^{\infty} F_{n-m,m}(r) J_{m-n}(ar) r\, \mathrm{d}r.\end{aligned}$$

$\delta_{p,q}$ is equal to 1 if $p = q$ and zero otherwise, and there is no summation over indices in the last term above.

The fact that this Fourier transform matrix, $\hat{f}_{mn}$, is inverted using the inversion formula to reconstruct $f(g)$ is seen as follows:

$$\begin{aligned} f(g) &= \int_0^\infty \text{trace}(\hat{f}(a)U_a(g))a\,\mathrm{d}a = \int_0^\infty \hat{f}_{mn}u_{nm}(g,a)a\,\mathrm{d}a \\ &= \int_0^\infty \left(\mathrm{i}^{n-m}\int_{r=0}^\infty F_{n-m,m}(r)J_{m-n}(ar)r\,\mathrm{d}r\right)\left(\mathrm{i}^{m-n}\mathrm{e}^{-\mathrm{i}[m\theta+(n-m)\phi]}J_{m-n}(ar)\right)a\,\mathrm{d}a \\ &= \mathrm{e}^{-\mathrm{i}[m\theta+(n-m)\phi]}\int_0^\infty\left(\int_{r=0}^\infty F_{n-m,m}(r)J_{m-n}(ar)r\,\mathrm{d}r\right)J_{m-n}(ar)a\,\mathrm{d}a. \end{aligned}$$

From here, the inversion is exactly the same as the *Hankel transform* [28], i.e. if

$$\hat{\phi}(a) = \int_0^\infty \phi(r)J_\nu(ar)r\,\mathrm{d}r$$

then

$$\phi(r) = \int_0^\infty \hat{\phi}(a)J_\nu(ar)a\,\mathrm{d}a.$$

That is, the Hankel transform is its own inverse. In our case, $\hat{\phi}(a) = \hat{F}_{n-m,m}(a)$, $\phi(r) = F_{n-m,m}(r)$, and $\nu = m - n$. This means that

$$\int_0^\infty \hat{f}_{mn}u_{nm}(g,a)a\,\mathrm{d}a = \mathrm{e}^{-\mathrm{i}[m\theta+(n-m)\phi]}F_{n-m,m}(r) = F_{jk}(r)\mathrm{e}^{-\mathrm{i}j\phi}\mathrm{e}^{-k\theta} = f(g).$$

This last step was simply a renaming of variables: $k = m$, and $n - m = j$. Since the sums over n and m are over $\{-\infty,\ldots,\infty\}$ there is no problem in doing this.

4.3. Parseval's equality

We now present a proof of Parseval's equality (also called the Plancherel formula) so that we have a mechanism for regularizing integral equations on $SE(2)$ using the Fourier transform. The proof presented here is similar to the one found in [29].

We begin by defining

$$f^*(g) = \overline{f(g^{-1})} \in \mathcal{L}^2(SE(2)) \qquad \forall\ f \in \mathcal{L}^2(SE(2)).$$

Then

$$h(g_x) = f * f^*(g_x) = \int_{SE(2)} f(g_\xi)f^*(g_\xi^{-1}\circ g_x)\,\mathrm{d}\mu(g_\xi) = \int_{SE(2)} f(g_\xi)\overline{f(g_x^{-1}\circ g_\xi)}\,\mathrm{d}\mu(g_\xi).$$

Evalulating this at the identity element $g_x = e$,

$$h(e) = \int_{SE(2)} f(g_\xi)\overline{f(g_\xi)}\,\mathrm{d}\mu(g_\xi) = \int_{SE(2)} |f(g)|^2\,\mathrm{d}\mu(g).$$

The function $h(g)$ can also be expressed via the inversion formula as

$$h(g) = \int_0^\infty \text{trace}(\hat{h}(a)U_a(g))a\,\mathrm{d}a.$$

Evaluated at the identity element, $U_a(e)$ is the identity operator, and so

$$h(e) = \int_0^\infty \text{trace}(\hat{h}(a))a\,\mathrm{d}a = \int_0^\infty \text{trace}(\hat{f}^*(a)\hat{f}(a))a\,\mathrm{d}a.$$

But

$$\hat{f}^*_{mn} = \int_{SE(2)} \overline{f(g^{-1})} u_{mn}(g^{-1}, a)\, \mathrm{d}\mu(g) = \int_{SE(2)} \overline{f(g)} u_{mn}(g, a)\, \mathrm{d}\mu(g).$$

This follows from the fact that $SE(2)$ is a unimodular group (possessing a volume element which is left and right invariant) and thus for any function $F \in \mathcal{L}^2(SE(2))$

$$\int_{SE(2)} F(g^{-1})\, \mathrm{d}\mu(g) = \int_{SE(2)} F(g)\, \mathrm{d}\mu(g).$$

From equation (9) we can then write

$$\hat{f}^*_{mn} = \int_{SE(2)} \overline{f(g) u_{nm}(g^{-1}, a)}\, \mathrm{d}\mu(g) = \overline{\hat{f}}_{nm} = \overline{\hat{f}}^T_{mn} = (\hat{f}_{mn})^*$$

i.e. $\hat{f}^* = (\hat{f})^*$ is the complex conjugate transpose of the Fourier transform matrix $\hat{f}$. This means that

$$h(e) = \int_0^\infty \mathrm{trace}((\hat{f})^*(a)\hat{f}(a)) a\, \mathrm{d}a = \int_0^\infty \|\hat{f}(a)\|_2^2 a\, \mathrm{d}a.$$

Equating the two expressions for $h(e)$, we get Parseval's equality for $SE(2)$:

$$\int_{SE(2)} |f(g)|^2\, \mathrm{d}\mu(g) = \int_0^\infty \|\hat{f}(a)\|_2^2 a\, \mathrm{d}a.$$

This relationship will be extremely useful when we regularize and solve equation (1) in section 6. But first, we examine operational properties, and a closed-form solution is presented in section 5.

4.4. Operational properties

Recall that the Fourier transform of the derivative of a square integrable function on the real line is defined using the inversion formula in equation (6) by differentiating under the integral

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{\infty} \hat{f}(\omega) u(x, \omega)\, \mathrm{d}\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \frac{\partial u}{\partial x}\, \mathrm{d}\omega. \tag{12}$$

Since $\frac{\partial u}{\partial x} = \mathrm{i}\omega u(x, \omega)$, we get that the Fourier transform of $\mathrm{d}f/\mathrm{d}x$ is $\mathrm{i}\omega\hat{f}(\omega)$ where $\hat{f}(\omega)$ is the Fourier transform of $f(x)$.

The exact same argument can be used to generate the Fourier transform matrices of derivatives of functions $f(g)$ where $g \in SE(2)$. For example, using the the inverse transform given in section 3 we get

$$r \frac{\partial f}{\partial r} = r \int_0^\infty \hat{f}_{nm} \frac{\partial u_{mn}}{\partial r} a\, \mathrm{d}a. \tag{13}$$

Since $u_{mn}(g(r, \phi, \theta), a) = \mathrm{i}^{n-m} \mathrm{e}^{-\mathrm{i}[n\theta + (m-n)\phi]} J_{n-m}(ar)$, one gets $\partial u_{mn}/\partial r = \mathrm{i}^{n-m} \mathrm{e}^{-\mathrm{i}[n\theta+(m-n)\phi]}\, J'_{n-m}(ar)a$. Integrating equation (13) by parts, and assuming $\hat{f}_{nm}(a)$ is well behaved

$$\begin{aligned} r \int_0^\infty \hat{f}_{nm}(a) J'_{n-m}(ar) a^2\, \mathrm{d}a &= -\int_0^\infty \frac{d}{da}(a^2 \hat{f}_{nm}(a)) J_{n-m}(ar)\, \mathrm{d}a \\ &= -\int_0^\infty (2\hat{f}_{nm}(a) + a\hat{f}'_{nm}(a)) J_{n-m}(ar) a\, \mathrm{d}a. \end{aligned}$$

This means that

$$r\frac{\partial f}{\partial r} = -\int_0^\infty (2\hat{f}_{nm} + a\hat{f}'_{nm})u_{mn}a\,\mathrm{d}a$$

and so

$$\mathcal{F}\left(r\frac{\partial f}{\partial r}\right) = -\left(2\hat{f} + a\frac{\mathrm{d}\hat{f}}{\mathrm{d}a}\right). \tag{14}$$

We may then use Parseval's equality together with integration by parts to show that

$$\int_{SE(2)} \left| r\frac{\partial f}{\partial r}\right|^2 \mathrm{d}\mu(g) = \int_0^\infty \left\| a\frac{\mathrm{d}\hat{f}}{\mathrm{d}a}\right\|_2^2 a\,\mathrm{d}a.$$

We find that the transforms of the derivatives of $f(g)$ with respect to ϕ and θ are

$$\mathcal{F}\left(\frac{\partial f}{\partial \theta}\right) = \mathbf{A}\hat{f} \tag{15}$$

and

$$\mathcal{F}\left(\frac{\partial f}{\partial \phi}\right) = \hat{f}\mathbf{A} - \mathbf{A}\hat{f} \tag{16}$$

where $\mathbf{A}_{mn} = -\mathrm{i}m\delta_{mn}$ (no sum) is a diagonal matrix. This comes from the fact that $\partial u_{mn}/\partial\theta = (-\mathrm{i}n)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}[n\theta+(m-n)\phi]}J_{n-m}(ar) = -\mathrm{i}nu_{mn}$ (no sum), and $\partial u_{mn}/\partial\phi = -\mathrm{i}(m-n)\mathrm{i}^{n-m}\mathrm{e}^{-\mathrm{i}[n\theta+(m-n)\phi]}J_{n-m}(ar) = -\mathrm{i}(m-n)u_{mn}$ (no sum).

The transforms of other derivatives follow from these by composition.

5. A closed-form example

We will now show how equation (1) can be solved exactly in certain instances using the Fourier transform of functions on the Euclidean group. The next section will discuss regularization to yield well behaved approximate solutions in those cases when exact solutions are not possible.

Taking the Fourier transform of both sides of equation (1), the original problem becomes

$$\hat{f}(a)\hat{k}(a) = \hat{h}(a)$$

and thus

$$\hat{f}(a) = \hat{h}(a)[\hat{k}(a)]^{-1}.$$

The desired function is then recovered using the inverse transform as

$$f(g) = \mathcal{F}^{-1}(\hat{h}(a)[\hat{k}(a)]^{-1}).$$

Questions that naturally follow are: (1) How are the infinite dimensional matrices $U_a(g)$, $U_a(g^{-1})$, $\hat{f}(a)$, $\hat{k}(a)$, and $\hat{h}(a)$ handled? (2) When is an exact solution possible, and what can be done when an exact solution is not possible?

The answer to (2) will be provided in section 6. The answer to (1) is that for bandlimited functions k and h (i.e. functions with a finite frequency spectrum), the Fourier transform matrix will be mostly zeros, with only a finite submatrix with non-zero elements.

To make the discussion concrete consider the following functions $h, k \in \mathcal{L}^2(SE(2))$:

$$\begin{aligned} h(g) &= H_1(r)(C_h + 2A_h\cos\theta) + 2B_hH_2(r)\cos\phi \\ k(g) &= K_1(r)(C_k + 2A_k\cos\theta) + 2B_kK_2(r)\cos\phi \end{aligned} \tag{17}$$

where $A_h, A_k, B_h, B_k, C_h, C_k \geqslant 0$ are given real numbers. These functions can be considered as first-order approximations to the density functions associated with two different halves of the discretely actuated manipulator arm discussed in section 2. When the coordinate axis normal to the base of the manipulator (pointing in the direction of the manipulator) is defined to be x_1, a plot of these density function resembles the one in figure 2 for appropriate choices of the constants and functions $H_i(r)$ and $K_i(r)$.

These functions are written in complex form as

$$h(g) = H_1(r)(C_h + A_h(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})) + B_h H_2(r)(\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi})$$
$$k(g) = K_1(r)(C_k + A_k(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})) + B_k K_2(r)(\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi}).$$

$H_1(r)$, $H_2(r)$, $K_1(r)$ and $K_2(r)$ are rapidly decreasing, bounded, piecewise continuous functions on $\mathbb{R}^+$.

Due to orthogonality, all of the elements of the Fourier transform matrices $\hat{h}$ and $\hat{k}$ are zero except the following blocks:

$$\hat{h}^{3\times3}(a) = \int_{SE(2)} h(g) U_a^{3\times3}(g^{-1})\,\mathrm{d}\mu(g) = \begin{pmatrix} \hat{h}_{-1,-1} & \hat{h}_{-1,0} & \hat{h}_{-1,1} \\ \hat{h}_{0,-1} & \hat{h}_{0,0} & \hat{h}_{0,1} \\ \hat{h}_{1,-1} & \hat{h}_{1,0} & \hat{h}_{1,1} \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_1(a) & 0 & 0 \\ \beta_1(a) & \kappa_1(a) & \beta_1(a) \\ 0 & 0 & \gamma_1(a) \end{pmatrix}$$
$$\hat{k}^{3\times3}(a) = \int_{SE(2)} k(g) U_a^{3\times3}(g^{-1})\,\mathrm{d}\mu(g) = \begin{pmatrix} \hat{k}_{-1,-1} & \hat{k}_{-1,0} & \hat{k}_{-1,1} \\ \hat{k}_{0,-1} & \hat{k}_{0,0} & \hat{k}_{0,1} \\ \hat{k}_{1,-1} & \hat{k}_{1,0} & \hat{k}_{1,1} \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_2(a) & 0 & 0 \\ \beta_2(a) & \kappa_2(a) & \beta_2(a) \\ 0 & 0 & \gamma_2(a) \end{pmatrix}$$

where

$$U_a^{3\times3}(g^{-1}) = \begin{pmatrix} u_{-1,-1}(g^{-1}, a) & u_{-1,0}(g^{-1}, a) & u_{-1,1}(g^{-1}, a) \\ u_{0,-1}(g^{-1}, a) & u_{0,0}(g^{-1}, a) & u_{0,1}(g^{-1}, a) \\ u_{1,-1}(g^{-1}, a) & u_{1,0}(g^{-1}, a) & u_{1,1}(g^{-1}, a) \end{pmatrix}$$
$$\gamma_1(a) = A_h \int_0^\infty H_1(r) J_0(ar) r\,\mathrm{d}r$$
$$\kappa_1(a) = C_h \int_0^\infty H_1(r) J_0(ar) r\,\mathrm{d}r$$
$$\beta_1(a) = -\mathrm{i}B_h \int_0^\infty H_2(r) J_1(ar) r\,\mathrm{d}r$$

and

$$\gamma_2(a) = A_k \int_0^\infty K_1(r) J_0(ar) r\,\mathrm{d}r$$
$$\kappa_2(a) = C_k \int_0^\infty K_1(r) J_0(ar) r\,\mathrm{d}r$$
$$\beta_2(a) = -\mathrm{i}B_k \int_0^\infty K_2(r) J_1(ar) r\,\mathrm{d}r$$

and $J_0(x)$ and $J_1(x)$ are respectively the zeroth- and first-order Bessel functions.

If for instance

$$H_1(r) = \mathrm{e}^{-b_h r^2} \qquad H_2(r) = r\mathrm{e}^{-c_h r^2} \qquad K_1(r) = \mathrm{e}^{-b_k r^2} \qquad K_2(r) = r\mathrm{e}^{-c_k r^2} \tag{18}$$

for $b_h, b_k, c_h, c_k > 0$, then the identity [18]

$$\int_0^\infty \mathrm{e}^{-ax^2} J_\nu(bx) x^{\nu+1}\,\mathrm{d}x = \frac{b^\nu}{(2a)^{\nu+1}}\mathrm{e}^{-b^2/4a}$$

for $a > 0$ is used to give closed-form solutions for the above integrals.

While the inverse of the matrix $\hat{k}$ is not defined, if we limit our search for solutions to those functions $f(g)$ for which only $\hat{f}^{3\times3}$ is non-zero, then this particular problem can be restricted so that only 3×3 matrices are considered, and

$$\hat{f}^{3\times3} = \hat{h}^{3\times3}[\hat{k}^{3\times3}]^{-1}.$$

This of course assumes that $\hat{k}^{3\times3}$ is invertible (which is the case in this example when γ_2 and κ_2 are nonzero). The desired function is then recovered using the inverse transform as

$$f(g) = \int_0^\infty \mathrm{trace}(\hat{f}^{3\times3}(a) U_a^{3\times3}(g)) a\,\mathrm{d}a$$

where $U_a^{3\times3}(g, a)$ is the part of the unitary matrix representation of $SE(2)$ that will multiply the nonzero block of the Fourier transform matrix $\hat{f}$, i.e.

$$U_a^{3\times3}(g) = \begin{pmatrix} u_{-1,-1}(g,a) & u_{-1,0}(g,a) & u_{-1,1}(g,a) \\ u_{0,-1}(g,a) & u_{0,0}(g,a) & u_{0,1}(g,a) \\ u_{1,-1}(g,a) & u_{1,0}(g,a) & u_{1,1}(g,a) \end{pmatrix}.$$

The inverse of $\hat{k}^{3\times3}$ for this example is

$$\begin{pmatrix} \gamma_2 & 0 & 0 \\ \beta_2 & \kappa_2 & \beta_2 \\ 0 & 0 & \gamma_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1/\gamma_2 & 0 & 0 \\ -\beta_2/\gamma_2\kappa_2 & 1/\kappa_2 & -\beta_2/\gamma_2\kappa_2 \\ 0 & 0 & 1/\gamma_2 \end{pmatrix}.$$

Therefore

$$\hat{f}^{3\times3} = \begin{pmatrix} \gamma_1/\gamma_2 & 0 & 0 \\ \beta_1/\gamma_2 - (\beta_2\kappa_1)/(\gamma_2\kappa_2) & \kappa_1/\kappa_2 & \beta_1/\gamma_2 - (\beta_2\kappa_1)/(\gamma_2\kappa_2) \\ 0 & 0 & \gamma_1/\gamma_2 \end{pmatrix}$$

and so

$$\begin{aligned}\mathrm{tr}(\hat{f}^{3\times3} U_a^{3\times3}) &= (\gamma_1/\gamma_2)(u_{-1,-1}(g,a) + u_{1,1}(g,a)) + (\kappa_1/\kappa_2)u_{0,0}(g,a) \\ &\quad + (u_{-1,0}(g,a) + u_{1,0}(g,a))(\beta_1/\gamma_2 - (\beta_2\kappa_1)/(\gamma_2\kappa_2)).\end{aligned}$$

From the definition of the matrix elements of the unitary representation of $SE(2)$ given in equation (8), we get

$$\begin{aligned}\mathrm{tr}(\hat{f}^{3\times3} U_a^{3\times3}) &= 2\cos\theta J_0(ar)(\gamma_1/\gamma_2) + J_0(ar)(\kappa_1/\kappa_2) \\ &\quad + 2\mathrm{i}\cos\phi J_1(ar)(\beta_1/\gamma_2 - (\beta_2\kappa_1)/(\gamma_2\kappa_2)).\end{aligned}$$

Therefore,

$$f(g) = 2F_1(r)\cos\theta + F_2(r) + 2F_3(r)\cos\phi$$

where

$$\begin{aligned}F_1(r) &= \int_0^\infty (\gamma_1(a)/\gamma_2(a)) J_0(ar) a\,\mathrm{d}a \\ F_2(r) &= \int_0^\infty (\kappa_1(a)/\kappa_2(a)) J_0(ar) a\,\mathrm{d}a \\ F_3(r) &= \mathrm{i}\int_0^\infty (\beta_1(a)/\gamma_2(a) - (\beta_2(a)\kappa_1(a))/(\gamma_2(a)\kappa_2(a))) J_1(ar) a\,\mathrm{d}a.\end{aligned}$$

Since $\beta_1(a)$ and $\beta_2(a)$ are purely imaginary, $F_3(r)$ is real (as are $F_1(r)$ and $F_2(r)$).

As with the solution of usual Fredholm integral equations of the first kind, this solution is not unique. Because we assumed the dimension of the non-zero block of $\hat{f}$ to be 3×3, this solution is not highly oscillatory, and limiting ourselves to a 3×3 block is in effect a means of regularization. The next section addresses issues in the regularization of equations of the form of equation (1) which do not have exact closed-form solutions.

6. Regularization of integral equations on $SE(2)$

In this section, the problem of regularizing ill-posed Fredholm integral equations of the first kind with convolution kernel on the Euclidean motion group is addressed. That is, for given $k(g)$ and $h(g)$ it may not be possible to invert the equation

$$(k * f)(g) = h(g)$$

exactly in order to find $f(g)$. This will be the case for instance when the rank of the Fourier transform matrix $\hat{k}(a)$ is less than that of $\hat{h}(a)$. In such cases, we need a method of restating the original problem so that an approximate solution results. One such method was illustrated in the introduction of this paper using Hermite functions. Another, less cumbersome method is discussed now.

We can restate the original problem as that of finding the real-valued function $f(g) \in \mathcal{L}^2(SE(2))$ which minimizes

$$C = \int_{SE(2)} \{[k * f(g) - h(g)]^2 + \epsilon[f(g)]^2\} \, \mathrm{d}\mu(g).$$

This is a generalization of the zeroth-order Tikhonov regularization [12] from the case of functions on $\mathbb{R}$ to functions on $SE(2)$. The value of the positive real number ϵ is used as a tradeoff between how accurate the solution is and how well behaved it is.

Since all the functions involved are real-valued and $|f(g)|^2 = [f(g)]^2$, Parseval's equality can be used to transform this cost function to

$$C = \int_0^\infty \{\|\hat{f}(a)\hat{k}(a) - \hat{h}(a)\|_2^2 + \epsilon\|\hat{f}(a)\|_2^2\} a \, \mathrm{d}a.$$

This equation can be minimized by finding the value of $\hat{f}(a)$ at each value of a that minimizes

$$c(\hat{f}(a)) = \|\hat{f}(a)\hat{k}(a) - \hat{h}(a)\|_2^2 + \epsilon\|\hat{f}(a)\|_2^2. \tag{19}$$

We therefore calculate:

$$\frac{\partial c}{\partial \hat{f}^*} = \hat{f}(\hat{k}\hat{k}^* + \epsilon 1) - \hat{h}\hat{k}^* = \mathbf{0}. \tag{20}$$

This equation is inverted to yield

$$\hat{f} = \hat{h}\hat{k}^*(\hat{k}\hat{k}^* + \epsilon 1)^{-1}.$$

As we saw in the previous section, these matrices can be treated as finite-dimensional if the α and ϕ terms have finite frequencies. In this case it is possible to perform the inversion above, and invert the resulting Fourier transform matrix yielding the desired result:

$$f(g) = \int_0^\infty \mathrm{trace}(\hat{h}(a)\hat{k}^*(a)(\hat{k}(a)\hat{k}^*(a) + \epsilon 1)^{-1} U_a(g)) a \, \mathrm{d}a.$$

This approach will be demonstrated with an example in subsection 6.1. But first we generalize the approach so that not only an approximate solution can be generated, but that

one with desired smoothness properties is generated. This means changing our cost function to account for smoothness of $f(g)$.

To this end, consider the cost function

$$C = \int_{SE(2)} \left\{ [k * f(g) - h(g)]^2 + \epsilon [f(g)]^2 + \nu \left(r \frac{\partial f}{\partial r} \right)^2 + \sigma \left(\frac{\partial f}{\partial \phi} \right)^2 + \eta \left(\frac{\partial f}{\partial \theta} \right)^2 \right\} \mathrm{d}\mu(g). \tag{21}$$

The last three terms can be thought of as the magnitude squared of a weighted gradient of the function f. Using the operational rules derived in subsection 4.4, this is transformed to

$$C = \int_0^\infty c(\hat{f}(a), \hat{f}'(a), a)\, \mathrm{d}a$$

where

$$c = \left(\|\hat{f}(a)\hat{k}(a) - \hat{h}(a)\|_2^2 + \epsilon \|\hat{f}(a)\|_2^2 + \nu \left\| a \frac{\mathrm{d}\hat{f}}{\mathrm{d}a} \right\|_2^2 + \sigma \|\hat{f}\mathbf{A} - \mathbf{A}\hat{f}\|_2^2 + \eta \|\mathbf{A}\hat{f}\|_2^2 \right) a.$$

One way to solve this regularized problem is to expand each element of the Fourier transform matrix in a series with unknown coefficients, such as $\hat{f}_{mn} = \sum_{i=1}^{N} \alpha_{imn} \psi_{imn}(a)$, for an appropriate set of basis functions $\{\psi_{imn}\}$. The result is then integrated with respect to a, and the problem becomes a simple multivariable minimization in the variables α_{imn}.

Another approach is to use variational optimization (see e.g. [21]) to solve the problem exactly. Using this approach, necessary conditions for the cost function C to be minimized are given by the set of Euler–Lagrange equations

$$-\frac{\mathrm{d}}{\mathrm{d}a} \left(\frac{\partial c}{\partial \hat{f}'^*} \right) + \frac{\partial c}{\partial \hat{f}^*} = \mathbf{0}$$

(a $'$ denoting differentiation with respect to a) which yields a set of second-order differential equations to be solved with appropriate boundary conditions. For the cost function in (21) these equations are written explicitly as

$$-\frac{\nu}{a} \frac{\mathrm{d}}{\mathrm{d}a} \left(a^3 \frac{\mathrm{d}\hat{f}}{\mathrm{d}a} \right) + \hat{f}(\hat{k}\hat{k}^* + \epsilon 1) + \sigma(\hat{f}\mathbf{A}\mathbf{A}^* - \mathbf{A}\hat{f}\mathbf{A}^* - \mathbf{A}^*\hat{f}\mathbf{A} + \mathbf{A}^*\mathbf{A}\hat{f}) + \eta \mathbf{A}^*\mathbf{A}\hat{f} = \hat{h}\hat{k}^*. \tag{22}$$

The boundary conditions with which to solve these equations are

$$\lim_{a \to 0} a^3 \hat{f}'(a) = \mathbf{0} \qquad \text{and} \qquad \lim_{a \to \infty} a^n \hat{f}(a) = \mathbf{0} \qquad \forall\, n \in Z^+. \tag{23}$$

The first boundary condition is the free initial condition given by $\partial c / \partial \hat{f}'^* = \mathbf{0}$. The other end condition is a result of the fact that acceptable functions must be rapidly decreasing.

Coupled equations of this form can be solved numerically using standard numerical integration schemes. The inverse transform can also be inverted using numerical integration. In subsection 6.2 a closed-form perturbation solution is examined for a special case.

6.1. An example of zeroth-order Tikhonov regularization

Let us consider the closed-form example in section 5, where now $C_k = B_k = 0$. This means that

$$\hat{k}^{3\times 3}(a) = \begin{pmatrix} \gamma_2(a) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_2(a) \end{pmatrix}$$

which is singular. To compute a regularized solution, we first compute

$$(\hat{k}^{3\times3})^*[\hat{k}^{3\times3}(\hat{k}^{3\times3})^* + \epsilon 1]^{-1} = \begin{pmatrix} \frac{\gamma_2}{\gamma_2^2+\epsilon} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\gamma_2}{\gamma_2^2+\epsilon} \end{pmatrix}.$$

Assuming $\hat{h}$ as in section 5, we get

$$\hat{f}_\epsilon^{3\times3} = \hat{h}^{3\times3}(\hat{k}^{3\times3})^*[\hat{k}^{3\times3}(\hat{k}^{3\times3})^* + \epsilon 1^{3\times3}]^{-1} = \begin{pmatrix} \frac{\gamma_1\gamma_2}{\gamma_2^2+\epsilon} & 0 & 0 \\ \frac{\beta_1\gamma_2}{\gamma_2^2+\epsilon} & 0 & \frac{\beta_1\gamma_2}{\gamma_2^2+\epsilon} \\ 0 & 0 & \frac{\gamma_1\gamma_2}{\gamma_2^2+\epsilon} \end{pmatrix}.$$

Note that by regularizing in this way, the solution is made unique because

$$\hat{h}^{3\times3}(\hat{k}^{3\times3})^*[\hat{k}^{3\times3}(\hat{k}^{3\times3})^* + \epsilon 1^{3\times3}]^{-1} = [\hat{h}\hat{k}^*[\hat{k}\hat{k}^* + \epsilon 1]^{-1}]^{3\times3}$$

when $\hat{h}$ and $\hat{k}$ only have non-zero elements in the 3×3 block. This also means that zeroth-order regularization dictates that $f_\epsilon^{3\times3}$ contains the only non-zero elements of the regularized solution for the Fourier transform $\hat{f}_\epsilon$.

Using the Fourier inversion formula, we then get the regularized solution

$$f_\epsilon(g) = 2\cos\theta \int_0^\infty \frac{\gamma_1(a)\gamma_2(a)}{\gamma_2^2(a)+\epsilon} J_0(ar)a\,\mathrm{d}a + 2\mathrm{i}\cos\phi \int_0^\infty \frac{\beta_1(a)\gamma_2(a)}{\gamma_2^2(a)+\epsilon} J_1(ar)a\,\mathrm{d}a.$$

Since β_1 is purely imaginary, this result is real. Note that as $\epsilon \to 0$, $f_\epsilon \to f$ where f is the exact solution presented in section 5.

6.2. An example of first-order Tikhonov regularization

Let us consider the case when $\hat{h}$ and $\hat{k}$ are as they were in subsection 6.1. Furthermore, assume that $\epsilon = \eta = \sigma = 0$, and ν is a small positive number. The resulting Euler–Lagrange equations will be of the form

$$-\frac{\nu}{a}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3\frac{\mathrm{d}\hat{f}}{\mathrm{d}a}\right) + \hat{f}(\hat{k}\hat{k}^*) = \hat{h}\hat{k}^*$$

which is a special case of equation (22).

Since in this case all the entries in the matrices $\hat{h}$ and $\hat{k}$ are zero except those elements in the 3×3 blocks $\hat{h}^{3\times3}$ and $\hat{k}^{3\times3}$, one finds that

$$\frac{\nu}{a}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3\frac{\mathrm{d}\hat{f}_{mn}}{\mathrm{d}a}\right) = 0$$

for (m, n) outside of $[-1, 1] \times [-1, 1]$. Dividing by ν/a and integrating once yields: $a^3\,\mathrm{d}\hat{f}_{mn}/\mathrm{d}a = c_1$. However, we have from the boundary conditions (23) at zero that this quantity must be zero, and so $c_1 = 0$. Therefore $\mathrm{d}\hat{f}_{mn}/\mathrm{d}a = 0$, which is integrated to yield $\hat{f}_{mn} = c_2$. However, we have from the second boundary condition that $\hat{f}_{mn} \to 0$ as $a \to \infty$, which means that $c_2 = 0$, and so the only solution admitted for the Fourier transform matrix elements outside of the 3×3 block is the trivial solution.

The next step is to solve the remaining matrix differential equations:

$$-\frac{\nu}{a}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3\frac{\mathrm{d}\hat{f}^{3\times3}}{\mathrm{d}a}\right) + \hat{f}^{3\times3}(\hat{k}^{3\times3}(\hat{k}^{3\times3})^*) = \hat{h}^{3\times3}(\hat{k}^{3\times3})^*.$$

For this example there are three different kinds of scalar equations that result:

$$-\frac{\nu}{a}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3\frac{\mathrm{d}\hat{f}_{m,m}}{\mathrm{d}a}\right)+\gamma_1^2(a)\hat{f}_{m,m}=\gamma_1(a)\gamma_2(a)$$

$$-\frac{\nu}{a}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3\frac{\mathrm{d}\hat{f}_{0,m}}{\mathrm{d}a}\right)+\gamma_1^2(a)\hat{f}_{0,m}=\gamma_1(a)\beta_2(a)$$

$$-\frac{\nu}{a}\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3\frac{\mathrm{d}\hat{f}_{m,-m}}{\mathrm{d}a}\right)+\gamma_1^2(a)\hat{f}_{m,-m}=0$$

for $m=1,-1$ (no sum).

These uncoupled linear equations with variable coefficients are difficult to solve in closed form. In fact, even approximate solutions are difficult because the singular perturbation methods which are usually used for equations whose highest derivative is multiplied by a small parameter break down when both of the highest derivatives are multiplied by this parameter [16].

Nonetheless, it suffices to say that to zeroth order in ν, solutions to these equations which obey the boundary conditions are

$$\hat{f}_{m.-m}\approx\gamma_2/\gamma_1 \qquad \hat{f}_{0,m}\approx\beta_2/\gamma_1 \qquad \hat{f}_{m,-m}\approx 0$$

for $m=1,-1$ (no sum) assuming that the decay of β_2 and γ_2 is faster than that of γ_1 as $a\to\infty$ and the ratio does not blow up anywhere.

Thus, to this order, the solution is exactly the same as the answer in the previous subsection.

7. Conclusions

The properties of the Fourier transform of scalar-valued functions on the Euclidean motion group were used to generate exact and regularized solutions of convolution integral equations involving functions on the Euclidean group. Because of the convolution theorem and operational properties of the Fourier transform on the Euclidean group, exact and regularized solutions of convolution-type equations are solved very efficiently. This is because the Fourier transform of a given square-integrable function on the Euclidean group is a matrix which is parametrized by a positive real scalar, and therefore solving convolution equations in the Fourier domain for the case of the Euclidean group is much less cumbersome than using Galerkin/least-squares techniques.

Acknowledgments

This work was made possible by the National Young Investigator Award IRI-9357738 from the Robotics and Machine Intelligence Program at the National Science Foundation, and Presidential Faculty Fellow Award IRI-9453373. Special thanks go to Ms Imme Ebert-Uphoff for preparation of the figures in section 2, and to Dr Alexander Kyatkin and the anonymous reviewers for proofreading this text.

References

[1] Blackmore D and Leu M C 1992 Analysis of swept volume via Lie groups and differential equations *Int. J. Robotics Res.* **11** 516–37

[2] Chirikjian G S 1995 Kinematic synthesis of mechanisms and robotic manipulators with binary actuators *ASME J. Mech. Design* **117** 573–80
[3] Davis P J and Rabinowitz P 1975 *Methods of Numerical Integration* (New York: Academic)
[4] Diaconis P and Rockmore D 1990 Efficient computation of the Fourier transform on finite groups *J. Am. Math. Soc.* **3** 297–332
[5] Dooley A H 1989 A nonabelian version of the Shannon sampling theorem 1989 *SIAM J. Math. Anal.* **20** 624–33
[6] Ebert-Uphoff I and Chirikjian G S 1996 Inverse kinematics of discretely actuated hyper-redundant manipulators using workspace density *Proc. 1996 IEEE Int. Conf. on Robotics and Automation (Minneapolis, MN, April 1996)* pp 139–45
[7] Ebert-Uphoff I and Chirikjian G S 1995 Efficient workspace generation for binary manipulators with many actuators *J. Robotic Syst.* 383–400
[8] Ebert-Uphoff I and Chirikjian G S 1996 Discretely actuated manipulator workspace generation by closed-form convolution *Proc. 1996 ASME Mechanisms Conf. (Irvine, CA, August 1996)* in press
[9] Flanders H 1989 *Differential Forms with Applications to the Physical Sciences* (New York: Dover)
[10] Folland G B 1995 *A Course in Abstract Harmonic Analysis* (Ann Arbor, MI: Chemical Rubber Company)
[11] Fulton W and Harris J 1991 *Representation Theory: A First Course* (New York: Springer)
[12] Groetsch C W 1984 *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind* (Boston, MA: Pitman)
[13] Halmos P 1965 *Measure Theory* (Princeton, NJ: Van Nostrand)
[14] Hansen P C 1992 Numerical tools for analysis and solution of Fredholm integral equations of the first kind *Inverse Problems* **8** 849–72
[15] Hochstadt H 1973 *Integral Equations* (New York: Wiley)
[16] Kervorkian J and Cole J D 1981 *Perturbation Methods in Applied Mathematics* (New York: Springer)
[17] Kumahara K 1976 Fourier transforms on the motion group *J. Math. Soc. Japan* **28** 18–32
[18] Lebedev N N 1972 *Special Functions and their Applications* translated by R A Silverman (New York: Dover)
[19] Lenz R 1990 *Group Theoretical Methods in Image Processing* (Berlin: Springer)
[20] Linz P 1994 A new numerical method for ill-posed problems *Inverse Problems* **10** L1–6
[21] Lovelock D and Rund H 1989 *Tensors, Differential Forms, and Variational Principles* (New York: Dover)
[22] Mackey G W 1950 Functions on locally compact groups *Bull. Am. Math. Soc.* **56** 385–412
[23] Maslen D K 1993 Fast transforms and sampling for compact groups *PhD Dissertation* Department of Mathematics, Harvard University
[24] Miller W 1964 Some applications of the representation theory of the Euclidean group in three-space *Commun. Pure Appl. Math.* **17** 527–40
[25] Nachbin L 1965 *The Haar Integral* (Princeton, NJ: Van Nostrand)
[26] Orihara A 1961 Bessel functions and the Euclidean motion group *Tohoku Math. J.* **13** 66–71
[27] Rockmore D N 1994 Efficient computation of Fourier inversion for finite groups *J. Assoc. Comput. Mach.* **41** 31–66
[28] Sneddon I N 1995 *Fourier Transforms* (New York: Dover)
[29] Sugiura M 1990 *Unitary Representations and Harmonic Analysis* 2nd edn (Amsterdam: North-Holland)
[30] Vilenkin N J and Klimyk A U 1991 *Representation of Lie Groups and Special Functions* vols 1–3 (Dordrecht: Kluwer)
[31] Warner F W 1983 *Foundations of Differentiable Manifolds and Lie Groups* (New York: Springer)