DE-Vol. 65-2, Advances in Design Automation – Volume 2
ASME 1993

CLOSED-FORM PRIMITIVES FOR GENERATING VOLUME PRESERVING DEFORMATIONS

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Abstract

In this paper, methods for generating closed-form expressions for locally volume preserving deformations of general volumes in three dimensional space are introduced. These methods potentially have applications to computer aided geometric design, the mechanics of materials, and realistic real-time simulation and animation of physical processes. In mechanics, volume preserving deformations are intimately related to the conservation of mass. The importance of this fact manifests itself in design, and in the realistic simulation of many physical systems. Whereas volume preservation is generally written as a constraint on equations of motion in continuum mechanics, this paper develops a set of physically meaningful basic deformations which are intrinsically volume preserving. By repeated application of these primitives, an infinite variety of deformations can be written in closed form.

1 Introduction

This paper develops and ennumerates deformations of objects which locally preserve volume. Given a three dimensional object described in Cartesian coordinates $\vec{x} = [x_1, x_2, x_3]^T$, a deformation maps these coordinates into a new set of coordinates: $\vec{X} = \vec{X}(\vec{x})$. A volume preserving deformation is one for which the volume of the object is kept constant after the deformation. This is written mathematically as:

$$\int_{\vec{x}} dx_1 dx_2 dx_3 = \int_{\vec{X}} dX_1 dX_2 dX_3 = \int_{\vec{x}} \det(\nabla_{\vec{x}} \vec{X}) dx_1 dx_2 dx_3. \quad (1)$$

where

$$\nabla_{\vec{x}} \vec{X} = \begin{pmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{pmatrix}. \quad (2)$$

The above quantity is called the deformation gradient. Throughout this paper the subscript $\vec{x}$ is often dropped, and the gradient is written simply as ∇ when there is no ambiguity.

A *locally* volume preserving deformation is one for which

$$\det(\nabla_{\vec{x}} \vec{X}) = 1. \quad (3)$$

A locally volume preserving deformation is also globally volume preserving, i.e, Equation (1) is satisfied automatically by Equation (3), but the converse is not generally true. Furthermore, compositions of locally volume preserving deformations are also locally volume preserving. That is, given two locally volume preserving deformations: $\vec{F}(\vec{x})$ and $\vec{G}(\vec{x})$, the compositions $\vec{F}(\vec{G}(\vec{x}))$ and $\vec{G}(\vec{F}(\vec{x}))$ are also locally volume preserving. This is a direct result of the chain rule, e.g.:

$$\nabla_{\vec{x}} \vec{F}(\vec{G}(\vec{x})) = \nabla_{\vec{y}} \vec{F}(\vec{y}) \nabla_{\vec{x}} \vec{G}(\vec{x}), \quad (4)$$

(where $\vec{y} = \vec{G}(\vec{x})$) and the fact that the determinant of a product of matrices is the product of the determinants, so

$$\det(\nabla_{\vec{x}} \vec{F}(\vec{G}(\vec{x}))) = \det(\nabla_{\vec{y}} \vec{F}(\vec{y})) \det(\nabla_{\vec{x}} \vec{G}(\vec{x})) = 1 \cdot 1 = 1. \quad (5)$$

Locally volume preserving deformations have significance in solid mechanics, biomechanics, and solid geometric modeling because of their intimate relationship to the conservation of mass of incompressible materials. For instance, in the analysis and simulation of the large deflections of many nonlinear elastic and plastic materials, Equation (3) is incorporated as a constraint. Whether a solution is sought using analytical techniques or numerical techniques such as finite element methods, this constraint often arises. It is also the case that when one seeks to simulate the physical world in a realistic way, such constraints must be accounted for. Having a method of parametrizing classes of constant volume deformations could provide designers in mass-sensitive fields, such as aerospace engineering, a valuable tool for enumerating changes to current designs.

Many works in the computer graphics, mechanics, and geometric design literature have dealt with the deformation of solid models. In [Ba81], angle-preserving deformations of superquadric surfaces where used to generate a wide variety of forms. Approximate methods for enforcing volume preservation where also examined.

[Ba84] extended these ideas to include general local and global deformations of arbitrary volumes. [SeP86] developed free-form deformations of solid models based on trivariate Bernstein polynomials. [SeF86] looked at the particular case of volume preserving deformations within this framework. It was found that volume preserving deformations based on cubic polynomials are limited to simple shear and scaling, and compositions of the above. Free-form deformations are also considered in [GuO90]. [PlB88] developed methods for representing the deformations of general solids based on continuum mechanics. In this approach, as is commonly done in solid mechanics, volume preservation is represented as a constraint which is imposed by using Lagrange multipliers [LaRK78,Mal69].

Other works have considered the importance of volume preservation when simulating and analyzing biological systems. However, no general framework for describing volume preserving deformations has been developed. In [Mi88], the locomotion of snakes and worms is simulated. This includes simultaneous longitudinal contraction and radial dilation so as to preserve volume. In [ChHP89] deformable figures are defined and made to look as real as possible (including volumetric constraints on contracting muscle). In [ArHDMR92], locally volume preserving deformations with constant deformation gradients were used to model cardiac mechanics. In [KieS85], the importance of local volume preservation in 'muscular-hydrostats' is stated, but not analyzed.

Whereas volume preservation is generally written as a constraint on equations of motion in continuum mechanics, this paper develops a set of physically meaningful basic deformations which are locally volume preserving. By repeated application of these primitives, an infinite variety of deformations can be written in closed form without the need for constraint equations. This opens up the possibility for more realistic real time simulation and animation of the physical world.

In Section 2, physically intuitive 'Cartesian' deformations are defined and illustrated. In Section 3, two types of bending deformations based on planar offset curves are defined and used. Section 4 shows examples of how combinations of these closed form primitives can be used to generate more complicated locally volume preserving deformations.

2 Cartesian Deformations

In this section several locally volume preserving deformations which preserve parallelism between planar sections are examined. These are referred to here as Cartesian deformations since they are essentially motions of flat planar sections.

In Subsection 2.1, pure shear deformations are examined. In Subsection 2.2, pure 'twist' deformations are examined. In Subsection 2.3, stretching and contraction are formulated.

2.1 Simple Shear

A simple shear deformation is one for which planar segments slide over each other without any rotation or change of their normal (denoted by the vector $\vec{n}$). This deformation is expressed as :

$$\vec{S}(\vec{x}) = \vec{x} + d(\vec{x}\cdot\vec{n})\vec{t}(\vec{x} \quad \vec{n}) \tag{6}$$

where $\vec{t}$ is any vector defined such that $\vec{n}\cdot\vec{t}=0$, and $\vec{t}\cdot\vec{t}=1$. If one imagines that $\mathbf{R}^3$ is composed of an infinite number of parallel planes, each with normal $\vec{n}$, this deformation slides each of these planes a distance d in the $\vec{t}$ direction. Since $\vec{t}$ lies in the plane, the effect is that each plane is translated within itself.

As an example, consider the shear deformation:

$$\vec{S}_1(\vec{x}) = \begin{pmatrix} x_1 + d(x_2) \\ x_2 \\ x_3 \end{pmatrix}. \tag{7}$$

This corresponds to Equation (6) for the choice, $\vec{n}=\vec{e}_2$ and $\vec{t}=\vec{e}_1$.

By taking the partial derivatives $\frac{\partial \vec{S}}{\partial x_i}$ for $i=1,2,3$, and computing the triple product of these vectors (which is the determinant of the deformation gradient), one finds it equal to unity, no matter what choice for d is used. Figure 1 shows an example of a square before and after a shear deformation. In this case, $d(x_2)=x_2^2$.

As in all figures throughout this paper, the undeformed object is the square defined by the Cartesian product: $[-1,1]\times[-1,1]$. Each of the deformed squares is viewed within a large square: $[-2,2]\times[-2,2]$.

A useful extension of simple shear is 'fiber' shear. That is, instead of a whole plane translating, lines are translated independently of other parallel lines. An example is:

$$\vec{S}_2(\vec{x}) = \begin{pmatrix} x_1 + d(x_2,x_3) \\ x_2 \\ x_3 \end{pmatrix}. \tag{8}$$

An easy way to think of this deformation is that it is like the motion of uncooked spaghetti being removed from its box. Each strand can translate while remaining parallel to other strands.

2.2 Twisting

This deformation is similar to the simple shear in that three dimensional space is viewed as an infinite cascade of parallel plane sections. In this deformation, each plane is rotated about an axis which intersects the origin and is parallel to the normal. The plane is thus mapped back into itself. This is written as:

$$\vec{T}(\vec{x}) = (\vec{x}\cdot\vec{n})\vec{n} + \mathrm{ROT}[\vec{n},\alpha(\vec{x}\cdot\vec{n})](\vec{x}-(\vec{x}\cdot\vec{n})\vec{n}). \tag{9}$$

The notation $\mathrm{ROT}[\vec{n},\alpha]$ represents the rotation matrix which rotates vectors about the unit vector $\vec{n}$ by an angle α in accordance with the right hand rule. In this case α is a function of the distance along the axis of rotation, so the resulting deformation is a twist. One can verify by direct calculation that this is also a locally volume preserving deformation.

In this case, the axis of rotation is the same for each planar segment. It is an axis with direction $\vec{n}$ which passes through the origin of the coordinate system in which $\vec{x}$ is defined. Compositions of pure twist and shear along the same axis allows arbitrary translation and rotation of parallel planar segments.

2.3 Elongation and Contraction

This subsection again formulates deformations in which three dimensional space is viewed as an infinite cascade of parallel planes. Only now, each of these planes is translated along the normal direction instead of orthogonal to it. Furthermore, as material elements which occupy these planes are stretched or contracted in the normal direction, inverse operations must be performed in each of the planes. That is, to locally conserve volume, an element which is stretched in one direction must contract in an orthogonal direction. A functional relationship which satisfies this criteria is given by:

$$\vec{E}(\vec{x}) = f(\vec{x}\cdot\vec{n})\vec{n} + \frac{g(\vec{x}\cdot\vec{n})}{f'(\vec{x}\cdot\vec{n})}\left[\vec{t}\cdot(\vec{x}-(\vec{x}\cdot\vec{n})\vec{n})\right]\vec{t} \qquad (10)$$

$$+\frac{1}{g(\vec{x}\cdot\vec{n})}\left[(\vec{n}\times\vec{t})\cdot(\vec{x}-(\vec{x}\cdot\vec{n})\vec{n})\right](\vec{n}\times\vec{t})$$

where $f(\cdot), f'(\cdot), g(\cdot) > 0$, and $\vec{t}\cdot\vec{n} = 0$, where $\vec{t}\cdot\vec{t} = 1$. Otherwise, the functions $f(\cdot)$, $g(\cdot)$, and the vector $\vec{t}$ are arbitrary. Note that a $'$ represents differentiation of a function of a single variable.

Two special cases of the above are when $g(\cdot) = \sqrt{f'}(\cdot)$, and $g(\cdot) = 1$. In the first case, Equation (10) reduces to:

$$\vec{E}_1(\vec{x}) = f(\vec{x}\cdot\vec{n})\vec{n} + \frac{\vec{x}-(\vec{x}\cdot\vec{n})\vec{n}}{\sqrt{f'(\vec{x}\cdot\vec{n})}}. \qquad (11)$$

As an illustration of the second case, when $\vec{n} = \vec{e}_1$ and $\vec{t} = \vec{e}_2$, we get:

$$\vec{E}_2(\vec{x}) = \begin{pmatrix} f(x_1) \\ \frac{x_2}{f'(x_1)} \\ x_3 \end{pmatrix}. \qquad (12)$$

An example of this is shown in Figure 2, where $f(x_1) = \frac{1}{2}x_1^2 + \frac{3}{2}x_1$.

3 Deformations Based on Offset Curves

This section develops a class of deformations based on the geometry of offset curves. Section 3.1 reviews basic properties of offset curves. Section 3.2 introduces pure bending deformations based on offset curves. Section 3.3 introduces the concept of offset shear-bending deformations.

3.1 Some Properties of Planar Offset-Curves

The offset of a planar 'backbone' (or generator) curve is a curve which is parallel to the backbone. This is intimately related to the envelope of a circle whose center is moving along a backbone curve. Applications of offset curves include planning the trajectories of numerically controlled milling machines [TiH84,Ph92], and the locomotion of snake-like robots [ChB91]. Properties of offset curves have been studied in [FaN90a,FaN90b]. In this subsection, some properties of offset curves are reviewed. These properties will be used in the following subsections to define locally volume preserving 'offset deformations.'

In the plane, an offset curve, $\vec{o}(L)$, of a given backbone curve, $\vec{c}(L)$, is defined as:

$$\vec{o}(L) = \vec{c}(L) + r_0\vec{n}(L) \qquad (13)$$

where $\vec{n}(L)$ is the unit normal to the curve $\vec{c}(L)$, and r_0 is a constant called the offset distance. For convenience, L is taken to be the arclength of the curve $\vec{c}(\cdot)$.

The set of offset curves of a given curve can be thought of as curves which are all parallel to each other with different values of r_0. The notion that two curves are parallel is a reflective property. That is, if 'A' is parallel to 'B,' then 'B' is parallel to 'A'. To see that offset curves are in fact parallel to each other is straight forward.

Suppose we take the offset of an offset curve as follows:

$$\vec{i}(L) = \vec{o}(L) + r_1\vec{m}(L) \qquad (14)$$

where r_1 is the offset distance of the second offset curve with respect to the first, and $\vec{m}(L)$ is the unit normal to $\vec{o}(L)$. By taking the derivative of (13) with respect to L and using the Frenet-Serret equations [MilP77] for the planar case, one finds that

$$\frac{d\vec{o}}{dL} = (1 - r\kappa)\vec{u}. \qquad (15)$$

where $\vec{u}(L)$ is the unit tangent vector to $\vec{c}(L)$. This means that the unit tangent to $\vec{o}(L)$, which is $\frac{d\vec{o}}{dL}/|\frac{d\vec{o}}{dL}|$, is the same as the unit tangent to $\vec{c}$. It follows trivially that they then have the same unit normal. Thus, $\vec{m}(L) = \vec{n}(L)$, and so,

$$\vec{i}(L) = \vec{c}(L) + (r_0 + r_1)\vec{n}(L). \qquad (16)$$

That is, the operation of taking an offset of an offset is the same as taking the offset of the original curve with offset distances equal to the sum of the individual offset distances. Likewise, some other arithmetic operations are easily performed on a set of offset curves.

Two interesting properties of sets of planar offset curves are presented in [FaN90a], and are restated as follows. First, the area between two offset curves of equal but opposite distance from a given backbone curve is invariant under changes in curvature of the original curve. Second, the sum of the lengths of two such offset curves is invariant under bending of the backbone curve.

The first of the above mentioned properties is relevant to the current discussion in that our goal is to model volume preserving deformations. However, the area preserving properties of regions enclosed by offset curves are *global*, not *local* properties. That is, the total area is preserved, but if one looks at small elements within an area bounded by offset curves, small area elements can change as the backbone curve geometry changes.

For instance, if one considers a rectangle of width w and height h, the area is $A = wh$. If the centerline of the area bends into a circular arc, the area will be preserved, provided the area does not overlap itself. This is observed by taking the difference in area of two concentric circles with radii differing by w and considering the portion of the resulting annular area which has a centerline of length equal to the original centerline. The area of the whole annulus is $A = \pi[(w+r)^2 - r^2]$. The centerline of the annular area is a third concentric circle with radius $r + w/2$ which bisects the annulus. Such a centerline will have circumference: $2\pi(r + w/2)$. The portion of the centerline of equal length as the original centerline is given by the ratio: $h/\pi(2r+w)$. The area of the segment of the annular area of centerline length h, is then

$$A = h/\pi(2r+w) \times \pi[(w+r)^2 - r^2] = hw.$$

However, the area elements on the inside of the circular arc will be compressed, while those on the outside will be stretched. Thus, this is not locally area preserving. As stated earlier, the area preserving properties associated with planar offset curves is generally a global, not local property. However, by extending the idea of an offset curve the following subsection develops two types of locally volume preserving deformations.

3.2 Bending Deformations which Locally Preserve Volume

This section develops a closed-form intrinsic parametrization of a class of bending deformations which are locally volume preserving. The following subsubsections develop two analytical models. These models are planar, though there are natural extensions to three dimensions. In Subsection 3.2.1 an analytical formulation based on 'variable offset' curves and the area contained within these geometric structures is investigated. In subsection 3.2.2, a deformation based on bending and reparametrization of a collection of constant offset curves is developed.

3.2.1 Variable Offset Bending

For a deformation to be locally area preserving, the area of each infinitesimal element must remain constant during the deformation. In order for the offset curve model to incorporate this feature, a generalized definition of a planar *variable offset bounded area* is defined below:

$$\vec{O}(\vec{x}) = \vec{c}(x_1) + r(x_1, x_2)\vec{n}(x_1). \tag{17}$$

This expression has a dual meaning. First, it can be considered as a deformation of a region in $x_1 - x_2$ space. Second, it is of the form of a set of offset curves with variable offset distance. Note that the parameter x_1 is not only a coordinate in the reference configuration, but also the arclength of the backbone curve. The above definition gives the function $r(x_1, x_2)$ two spatial degrees of freedom. The first is so that the offset can vary with the backbone curve parameter. The second is so that any point within the area bounded by two variable offset curves can be specified with the coordinates (x_1, x_2). If for instance, the initial backbone is a straight line $\vec{c} = x_1\vec{e}_1$, and $r(x_1, x_2) = x_2$, then initially (x_1, x_2) are the Cartesian coordinates for this slab, and they serve as referential coordinates for any continuously deformed configuration. See [LaRK78,Mal69] for definitions and details of referential descriptions of material deformation.

So that the model can incorporate the constraint of local volume (area) preservation no matter what kind of bending occurs, the function $r(x_1, x_2)$ is left undetermined for the time being. In order to enforce the local area preservation constraint, the following expression is observed:

$$\det\left(\nabla\vec{O}\right) = \left|\frac{\partial\vec{O}}{\partial x_1} \times \frac{\partial\vec{O}}{\partial x_2}\right| = 1. \tag{18}$$

That is, the area of each infinitesimal element defined in the Cartesian product of the two intervals: $[x_1, x_1 + dx_1] \times [x_2, x_2 + dx_2]$ must be independent of bending.

Evaluation of Equation (18) by substituting in (17), and observing:

$$\frac{\partial\vec{O}}{\partial x_1} = (1 - r\kappa)\vec{u} + \frac{\partial r}{\partial x_1}\vec{n} \quad ; \quad \frac{\partial\vec{O}}{\partial x_2} = \frac{\partial r}{\partial x_2}\vec{n}, \tag{19}$$

and using the fact that $\vec{u}\cdot\vec{u} = \vec{n}\cdot\vec{n} = 1$ and $\vec{u}\cdot\vec{n} = 0$, yields

$$(1 - r\kappa)\frac{\partial r}{\partial x_2} = 1. \tag{20}$$

This expression is integrated with respect to x_2 to yield:

$$(r - \frac{1}{2}\kappa r^2) = x_2 + c(x_1). \tag{21}$$

The arbitrary function $c(x_1)$ is taken to be zero. Note that a nonzero choice of $c(x_1)$ corresponds to composing a shear deformation, i.e., replacing x_2 with $x_2 + c(x_1)$ is a shear in the same way that Equations (6) and (7) are.

Using the quadratic formula to solve for $r(x_1, x_2)$, one finds that:

$$r(x_1, x_2) = \frac{1 \pm (1 - 2\kappa(x_1)x_2)^{\frac{1}{2}}}{\kappa(x_1)}, \tag{22}$$

of which the negative root is used. This is used because as $\kappa(x_1)$ goes to zero, $r(x_1, x_2)$ should converge to x_2, i.e., the unbent configuration should correspond to the slab parametrized with Cartesian coordinates (x_1, x_2). Note also that curvature of the backbone curve is always limited so that $x_2\kappa < \frac{1}{2}$ to avoid singularities.

As an example, consider the planar backbone curve:

$$\vec{c}(x_1) = \left(\frac{1}{a}\sin ax_1, \frac{1}{a}(1 - \cos ax_1)\right)^T \tag{23}$$

The deformed region is initially the square $[-1, 1] \times [-1, 1]$ shown in Figure 1(a). In effect, (17) and (22-23) define a deformation which bends the x_1 axis into a circular arc while preserving area locally. Figure 3 shows this for $a = \pi/12$.

3.2.2 Offset Shear-Bending

In this subsection, the properties of offset curves are exploited further. It is shown how allowing shear along directions parallel to the backbone curve preserves volume locally as the curve bends. This deformation is shown to be locally volume preserving when each of the collection of offset curves is parametrized by it's own arc length instead of the backbone curve arclength, i.e., a reparametrization is required. Shear in this context is achieved by simply allowing translation of each offset curve, while maintaining parallelism with the backbone curve.

Given a set of offset curves of the form:

$$\vec{O}(s, x_2) = \vec{c}(s) + x_2\vec{n}(s), \tag{24}$$

where the arc length measure of the backbone curve is denoted here as s, the arc length measure along each offset curve is:

$$x_1 = L(s, x_2) = \int_0^s \left|\frac{\partial\vec{O}}{\partial s}\right| ds = \int_0^s (1 - x_2\kappa(\sigma))d\sigma = s - x_2\theta(s). \tag{25}$$

This expression is rarely algebraicly invertible. That is, in general the relationship $s = \hat{s}(x_1, x_2)$ cannot be found in closed form. Nonetheless, such a relationship (even if it is not expressible in closed form) does exist.

If we choose the coordinates, x_1 and x_2, and reparametrize the set of offset curves such that

$$\hat{\vec{O}}(x_1, x_2) = \hat{\vec{O}}(L(s, x_2), x_2) = \vec{O}(s, x_2), \tag{26}$$

then $\hat{\vec{O}}(\vec{x})$ (viewed as a deformation) preserves local area independent of changes in curvature of the backbone curve. Proof of this fact is given below by direct calculation. The chain rule yields:

$$\frac{\partial\vec{O}}{\partial x_2} = \frac{\partial\hat{\vec{O}}}{\partial L}\frac{\partial L}{\partial x_2} + \frac{\partial\hat{\vec{O}}}{\partial x_2} \quad ; \quad \frac{\partial\vec{O}}{\partial s} = \frac{\partial\hat{\vec{O}}}{\partial x_1}\frac{\partial L}{\partial s}. \tag{27}$$

Putting Equations (27) in matrix form, and using the fact that $\frac{\partial L}{\partial s} = 1 - x_2\kappa(s)$ and $\frac{\partial L}{\partial x_2} = -\theta(s)$, we get:

$$\begin{pmatrix} \frac{\partial\vec{O}}{\partial s} \\ \frac{\partial\vec{O}}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 - x_2\kappa & 0 \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial\hat{\vec{O}}}{\partial x_1} \\ \frac{\partial\hat{\vec{O}}}{\partial x_2} \end{pmatrix}. \tag{28}$$

Inverting this equation, we find that:

$$\begin{pmatrix} \frac{\partial \dot{\vec{O}}}{\partial x_1} \\ \frac{\partial \dot{\vec{O}}}{\partial x_2} \end{pmatrix} = \frac{1}{1 - x_2\kappa} \begin{pmatrix} 1 & 0 \\ 0 & 1 - x_2\kappa \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{O}}{\partial s} \\ \frac{\partial \vec{O}}{\partial x_2} \end{pmatrix}. \qquad (29)$$

One then finds that:

$$\frac{\partial \hat{\vec{O}}}{\partial s} \times \frac{\partial \hat{\vec{O}}}{\partial x_2} = \frac{1}{1 - x_2\kappa} \left(\frac{\partial \vec{O}}{\partial x_1} \times \frac{\partial \vec{O}}{\partial x_2} \right) = \vec{u} \times \vec{n} = \vec{e}_3. \qquad (30)$$

Thus, independent of the curvature of the backbone curve, κ, local area is preserved using these deformations. Note however, that we must restrict ourselves to conditions under which $x_2\kappa < 1$ in order to avoid the singularities which occur when $1 = \kappa x_2$.

As an example, consider a backbone curve in Equation (23). In this case, $\kappa(s) = a$, and (25) can be inverted to yield: $s = x_1/(1 - ax_2)$. Figure 4 shows this type of deformation applied to the same referential square as used earlier.

4 Examples of Composition of Deformations

This section illustrates how composition of the primitive volume preserving deformations presented earlier in this paper can be used to efficiently generate an infinite variety of volumetric shapes from a single referential volume.

As a first example, consider a combined stretching and bending deformation. In this example, the stretch deformation in Equation (12) is first applied, then the bending deformation defined by Equations (17) and (22-23) is applied. The resulting deformation is:

$$\vec{C}_1(\vec{x}) = \vec{O}(\vec{E}_2(\vec{x})). \qquad (31)$$

The choice of primitive deformations resulting in Figure 5 are:

$$\vec{E}_2(\vec{x}) = \begin{pmatrix} \frac{1}{4}x_1^2 + \frac{3}{2}x_1 \\ \frac{x_2}{\frac{x_1}{2} + \frac{3}{2}} \\ x_3 \end{pmatrix} \qquad (32)$$

and

$$\vec{O}(\vec{x}) = \begin{pmatrix} \frac{1}{a}\sin ax_1 - r(x_1, x_2)\sin ax_1 \\ \frac{1}{a}(1 - \cos ax_1) + r(x_1, x_2)\cos ax_1 \\ x_3 \end{pmatrix}, \qquad (33)$$

where $r(x_1, x_2) = \frac{1-(1-2ax_2)^{\frac{1}{2}}}{a}$, and $a = \pi/12$.

Figure 6 is a composition of shear and the bending deformation based on variable offsets defined in Subsubsection 3.2.1. This is written as $\vec{O}(\vec{S}_1(\vec{x}))$.

Figure 7 shows a composition of shear and the type of bending described in Subsubsection 3.2.2. That is, this deformation is of the form: $\hat{\vec{O}}(\vec{S}_1(\vec{x}))$ where $\vec{S}_1(\cdot)$ is defined in Equation (7), and $\hat{\vec{O}}(\vec{x})$ was defined in Equation (33). In this figure, $d(x_2) = \frac{1}{2}x_2$.

Note the difference between Figures 6 and 7.

5 Conclusion

This paper has presented methods for generating and using locally volume preserving deformations which can be written in closed form. A combination of classical differential geometry and parametric geometry were used to generate these closed-form deformations. These deformations potentially have applications to computer aided geometric design when mass and volume constraints restrict design choices. Further applications may include efficient approximations to problems in the mechanics of materials and biomechanics. Similarly, the closed-form primitives presented here could be used to approximate the physical universe in such diverse areas as robotics (e.g., modeling the kinematics and dynamics of robots with flexible actuators), and even virtual reality (e.g., real-time simulation of virtual contact with a 'squishy' environment).

It should be noted that a wide variety of locally volume preserving deformations can also be generated using dynamical systems theory and/or fluid mechanics. However, closed form solutions are relatively rare in these fields.

6 References

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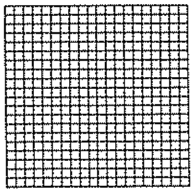

Figure 1(a): A Referential (Undeformed) Square

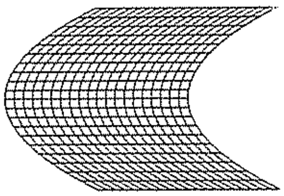

Figure 1(b): A Simple Shear Deformation : $\vec{S}_1(\vec{x})$

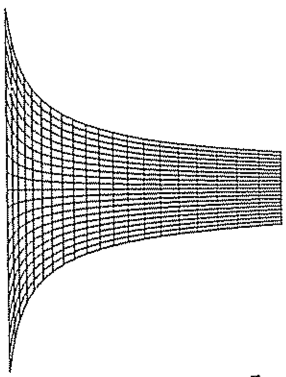

Figure 2: Nonuniform Stretching : $\vec{E}_2(\vec{x})$

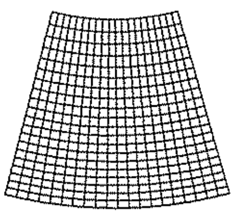

Figure 3: Variable-Offset Bending : $\vec{O}(\vec{x})$

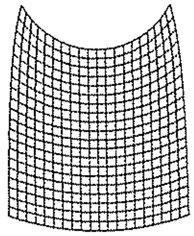

Figure 4: Offset Shear-Bending : $\tilde{\vec{O}}(\vec{x})$

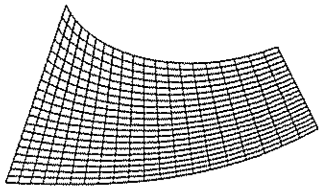

Figure 5: Stretch and Bending : $\vec{O}(\vec{E}_2(\vec{x}))$

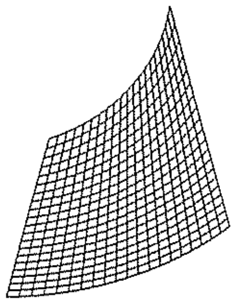

Figure 6: Shear and Variable-Offset Bending : $\vec{O}(\vec{S}_1(\vec{x}))$

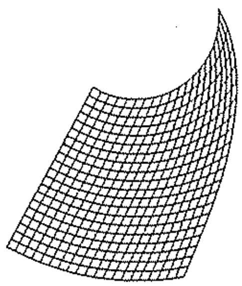

Figure 7: Shear and Offset Shear-Bending : $\tilde{\vec{O}}(\vec{S}_1(\vec{x}))$