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# NUMERICAL SYNTHESIS OF BINARY MANIPULATOR WORKSPACES USING THE FOURIER TRANSFORM ON THE EUCLIDEAN MOTION GROUP

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## ABSTRACT

In this paper we apply the Fourier transform on the Euclidean motion group to solve problems in kinematic design of binary manipulators. We begin by reviewing how the workspace of a binary manipulator can be viewed as a function on the motion group, and how it can be generated as a generalized convolution product. We perform the convolution of manipulator densities, which results in the total workspace density of a manipulator composed of double the number of modules. We suggest an anzatz function which approximates the manipulator's density in analytical form and has few free fitting parameters. Using the anzatz functions and Fourier methods on the motion group, linear and non-linear inverse problems (i. e. problems of finding the manipulator's parameters which produce the total desired workspace density) are solved.

## 1 INTRODUCTION

A robotic manipulator is generally constructed of rigid links and actuators, such as motors or hydraulic cylinders. For actuators with only a finite number of states, as is the case with stepper motors or pneumatic cylinders, the robotic arm has a finite number of configurations and can reach only a finite number of frames (positions and orientations). Each frame is completely determined by the position of its origin and orientation relative to a fixed frame, and thus is completely specified by an element of the motion group $SE(N)$ [2] (see (Murray et. al. 1994) for references on the motion group).

For discretely actuated manipulators the *workspace density*, which is defined as the number of reachable frames per unit volume of the motion group $SE(N)$ (Ebert-Uphoff and Chirikjian 1995), determines how accurately a position and orientation can be reached. This density information is important for the kinematic design of manipulators and for planning the motions of discretely actuated manipulator arms (Ebert-Uphoff and Chirikjian 1996a).

An important aspect of the manipulator design problem is to specify the density of reachable frames throughout the workspace. That is, areas which must be reached with great accuracy should have high density, and those areas of the workspace which are less important need less density. For relatively few actuators, the design problem may be solved by enumerating reachable frames (positions and orientations) and using an iterative procedure as discussed in (Chirikjian 1995). A grey scale density plot of the workspace density of a planar manipulator with six modules (the top of each module can attain eight possible positions and orientations relative to its base) with a total of $242,047$ possible configurations is depicted in Fig. (1) for the end-effector orientation angle $\theta = 0$.

In computing the workspace density $20,097$ "singular" configurations were not counted. The "singular" configurations appear when the neighbor modules overlap. This is possible when the condition $[\arccos((s^2 + l_{min}^2 - l_{max}^2)/(2\, l_{min}\, s)) + \arccos((2\, l_{min}^2 - s^2)/(2\, l_{min}^2))] \geq \pi$ is satisfied ($l_{min}$ and $l_{max}$ are the minimal and maximal lengths of actuators, $s$ is the length of the module base, see Fig. (3)).

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[2] $SE(N)$ denotes "special Euclidean group", which is the group of rigid motions of $N$-dimensional Euclidean space. We refer to this group simply as the "motion group".



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However, to compute the workspace density function using brute force and iterating is computationally intractable for a large number of actuated modules $n$. E.g., it requires $K^n$ evaluations of the kinematic equations relating actuator state to the resulting end frame for a manipulator with $n$ modules each with $K$ states.

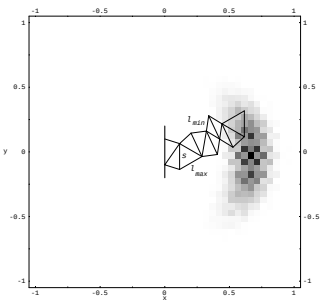


Fig. 1

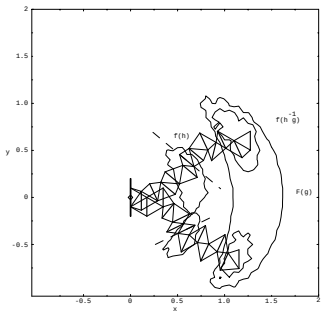
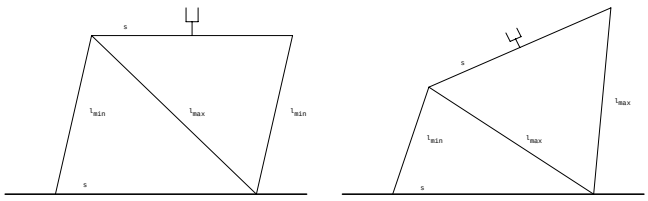



Fig. 2

However, as has been shown in (Ebert-Uphoff and Chirikjian 1996b), partitioning the manipulator into segments, computing the workspace density of each segment, and computing the whole workspace as a convolution of workspaces of each segment is a very efficient way to overcome this problem. Using the convolution approach allows us to reduce the exponential growth in $n$ to linear growth in $n$ for inequivalent segments and to $O(\log n)$ for identical segments.



a) b)

Fig. 3

The workspace density of a planar manipulator which has double the number of identical modules of one with workspace density $f(g)$ may be generated as a convolution on the motion group

$$F(g) = \int_{SE(2)} f(h)\, f(h^{-1} \circ g) d\mu(h) \ , \qquad (1)$$

where $d\mu(g)$ is an integration measure on the motion group (see section 2). We geometrically illustrate the convolution integral (1) in Fig. (2) as a sweeping procedure. This is explained in detail in (Chirikjian and Ebert-Uphoff 1998).

To calculate the density $F(g)$ of the "composed" manipulator we have to multiply the density of the lower part of the manipulator $f(h)$ by the density of the upper part $f(h^{-1} \circ g)$ (relative to the end-effector of the lower part) and integrate the product with respect to all possible intermediate configurations $h$ of the lower part end-effector.

However, a problem still exists with this approach, because the convolution itself is a very costly computation for fine discretizations. For problems of kinematic design of manipulators, where many convolutions may be needed for fitting the manipulator's workspace to the desired workspace density function, the convolutions may take days of computation time. Moreover, the inverse problem, i. e. the problem of fitting the manipulator parameters to the desired workspace density, may still be intractable by direct integration methods.

In analogy with Fourier methods on the real line, on the sphere, or on finite groups, which give an efficient way to perform convolutions and allow one to apply Fast Fourier Transform (FFT) methods (Elliot, Rao 1982), (Driscoll, Healy 1994), (Rockmore 1994), the application of Fourier methods on the motion group provides a considerable saving in the computation time. Fourier methods, moreover, allow us to solve (at least approximately) linear and nonlinear inverse problems, i. e. the problems of determining the manipulator's parameters which produce the total desired workspace density.

The mathematical framework for non-commutative Fourier methods on the two and three dimensional motion group with applications to the solution of convolution equations were developed in (Chirikjian 1996), (Kyatkin and Chirikjian 1996a; Kyatkin and Chirikjian 1996b). Here we implement these methods numerically for the two dimensional motion group and apply these methods to the generation of manipulator workspaces by the convolution method and for the solution of inverse problems. We also estimate the accuracy (in the sense of quadratic norm) of the Fourier transform methods.

In section 2 we give general references for the Fourier transform on the 2D motion group.The direct convolution of manipulator workspace densities performed by the Fourier method is described in section 3. The accuracy of the method and the required computation time are given.

In section 4 we suggest an anzatz function which describes the manipulator's density in an analytical form and has relatively few free parameters. Using anzatz functions and Fourier methods we solve the linear and non-linear inverse problems.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## 2 THE FOURIER TRANSFORM ON THE MOTION GROUP

Here we give briefly the general expressions which define the Fourier transform on the two dimensional motion group (for more complete references on the two dimensional case and for the three dimensional case see (Chirikjian 1996; Kyatkin and Chirikjian 1996a; Kyatkin and Chirikjian 1996b).

Each element of $SE(2)$ is parametrized in either rectangular or polar coordinates as:

$$g(r_1, r_2, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & r_1 \\ \sin\theta & \cos\theta & r_2 \\ 0 & 0 & 1 \end{pmatrix}$$

or

$$g(r, \phi, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & r\cos\phi \\ \sin\theta & \cos\theta & r\sin\phi \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $r = \mid \mathbf{r} \mid$. The group law is simply matrix multiplication.

We need to use the unitary representations of the motion group $SE(2)$ (see (Vilenkin, Klimyk 1991), (Talman 1968), (Vilenkin 1956), (Orihara 1961), (Sugiura 1990) for general definitions) to generate the Fourier transform on the two dimensional motion group. If we use these representations then the Fourier transform of the convolution of functions may be written as the matrix product of the Fourier transform of each function (see below), which gives considerable savings in computation time.

A number of works including (Vilenkin 1956), (Orihara 1961), (Talman 1968) have shown that the matrix elements of the unitary irreducible representation $\mathcal{U}(g,p)$ of $SE(2)$ are given by

$$u_{mn}(g(r,\phi,\theta),p) = i^{n-m} e^{-i[n\theta+(m-n)\phi]} J_{n-m}(p\,r) \qquad (2)$$

for $m$, $n \in \mathbf{Z}$, where $J_\nu(x)$ is the $\nu^{th}$ order Bessel function, $g \in SE(2)$, and $p$ is a continuous parameter which enumerates the representations, in analogy with the Fourier transform parameter on the real line.

From this expression, and the fact that $\mathcal{U}(g,p)$ is a unitary representation, we have that:

$$u_{mn}(g^{-1}(r,\phi,\theta),p) = u^{-1}_{mn}(g(r,\phi,\theta),p) =$$

$$\overline{u_{nm}(g(r,\phi,\theta),p)} = i^{n-m} e^{i[m\theta+(n-m)\phi]} J_{m-n}(pr). \qquad (3)$$

**Symmetry property.** The matrix elements are related by the symmetry

$$\overline{u_{mn}(g,p)} = (-1)^{m-n} u_{-m,-n}(g,p) \qquad (4)$$

The inner product of square-integrable functions on the motion group is given by

$$(f,h) = \int_{SE(2)} \overline{f(g)}\, h(g)\, d\mu(g) \quad . \qquad (5)$$

The invariant integration measure on $SE(2)$ is given by

$$d\mu(g(r,\phi,\theta)) = \frac{1}{(2\pi)^2} r\, dr d\phi d\theta$$

($r$, $\phi$ are the radial and angular parts of the translation vector $\mathbf{r}$ and $\theta$ is the $SO(2)$ orientation angle).

**Definition.** *For any integrable complex-valued function $f(g)$ on the motion group $G$ we define the Fourier transform as*

$$\mathcal{F}(f) = \hat{f}(p) = \int_G f(g)\, \mathcal{U}(g^{-1},p)\, d\mu(g)$$

*where $g \in G$.*

For $SE(2)$ the matrix elements of $\mathcal{U}(g^{-1},p)$ are given in (3).

The inverse Fourier transform is used to reconstruct a function from its Fourier transform as:

$$f(g) = \mathcal{F}^{-1}(\hat{f}) = \int_0^\infty \mathrm{Tr}(\hat{f}(p)\mathcal{U}(g,p))\, p\, dp\,. \qquad (6)$$

**Plancherel equality.** The Plancherel equality for square-integrable functions on the motion group $G = SE(2)$ is:

$$\int_G |f(g)|^2 d\mu(g) = \int_0^\infty ||\hat{f}(p)||_2^2\, p\, dp.$$

where $||\hat{f}(p)||_2^2$ is a Hilbert-Schmidt norm

$$||\hat{f}||_2^2 = \mathrm{Tr}(\hat{f}\,\hat{f}^\dagger) \quad ,$$

$\hat{f}^\dagger$ is a Hermitian conjugate of $\hat{f}$, and Tr is the trace.


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**Convolution property.** One of the most powerful properties of the Fourier transform of functions on $\mathbb{R}^N$ is that the Fourier transform of the convolution of two square-integrable functions is the product of the Fourier transforms of the functions. This property persists also for the convolution of functions on the motion group

$$(f_1 * f_2)(g) \;=\; \int_G f_1(h)\, f_2(h^{-1} \circ g)\, d\mu(h)\;, \qquad (7)$$

namely

$$\mathcal{F}(f_1 * f_2) \;=\; \mathcal{F}(f_2)\, \mathcal{F}(f_1), \qquad (8)$$

where now the product is a matrix product of the Fourier transform matrices

$$\mathcal{F}(f_1 * f_2)_{mn} = \sum_{k=-\infty}^{\infty} \hat{f}^2_{mk}\, \hat{f}^1_{kn}$$

where $\hat{f}^i_{mn} \;=\; \mathcal{F}(f_i)_{mn}$. We note that now the order of the product of Fourier transforms matters. In practice, this product is truncated at $k = \pm M$, where $M$ is a chosen finite number.

**Symmetries.** For the real function $f(g)$ we note a symmetry property of the Fourier transform, which follows from symmetry (4) of the matrix elements

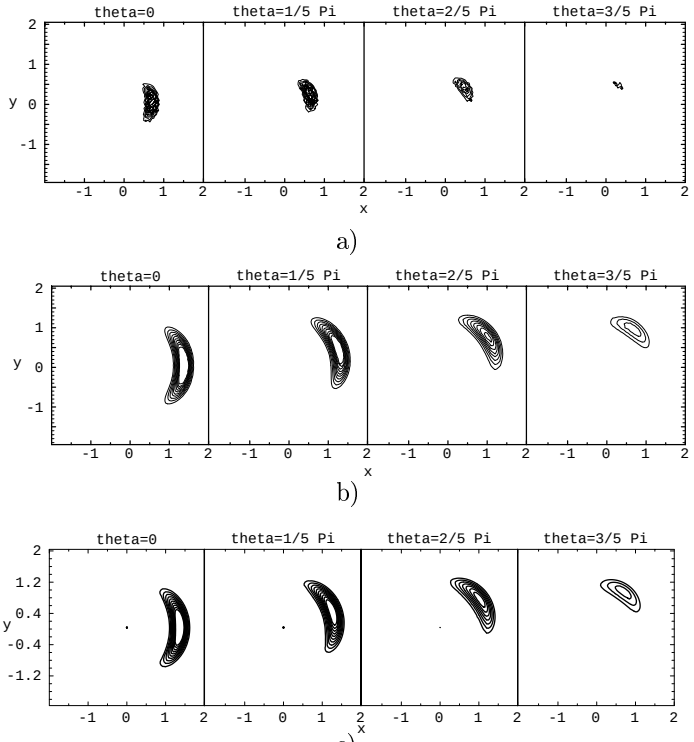
$$\overline{\hat{f}_{mn}} \;=\; (-1)^{(m-n)}\, \hat{f}_{-m,-n}\;. \qquad (9)$$

This symmetry is preserved under multiplication of Fourier transforms (which follows from the convolution property and the fact that the convolution of real-valued functions is real). This symmetry allows us to reduce the amount of computations by half.

## 3 APPLICATION OF FOURIER TRANSFORM METHODS TO WORKSPACE GENERATION OF DISCRETELY ACTUATED MANIPULATORS

Here we use Fourier methods on the motion group to efficiently calculate the workspace density function of a binary manipulator consisting of two segments (direct convolution problem). This density information is important for motion planning and kinematic design of binary manipulators. We also will apply Fourier methods to the solution of linear and non-linear inverse problems in the next section.

For a small number of modules $n$ ($n \approx 6-8$ for a two-dimensional manipulator with 3 discrete binary actuators in each module) the workspace density of a binary manipulator may be generated by brute force. However, the amount of computation increases exponentially in $n$ and for larger $n$ it cannot be performed by simply enumerating the states. As we mentioned before, partitioning the manipulator into segments, computing the workspace density of each segment (which may be done by brute force) and computing the whole workspace as a convolution of workspace densities of each segment is a very efficient way to overcome this problem (Ebert-Uphoff and Chirikjian 1996a). For identical segments an $O(K^n)$ computation is reduced to a very moderate $O(\log n)$ computation.




a) b) c)

Fig. 4

Below we apply Fourier methods on the motion group to efficiently compute the convolution of workspace densities of six module manipulator segments, depicted in Fig. (4a). It may be shown that the convolution of functions on the motion group $SE(2)$ may be performed in $O(N^{3/2})$ computations via Fourier transform, as opposed to $O(N^2)$ computations of the direct (on N-point grid in the coordinate space) integration.

We computed the convolution of workspace densities which corresponds to the workspace density of a twelve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module manipulator. The application of non-commutative Fourier methods gives a considerable saving in the computation time: while the convolution by direct integration required 14 hours of the computation time for our array size ($N^2 \approx 4 \cdot 10^9$), the Fourier methods allow us to compute the convolution ( generate workspace, compute the direct Fourier transform, perform matrix multiplication and the inverse Fourier transform) in around 33 minutes (for $M = 15$) and in 6.5 minutes (for $M = 4$) even for the large array size ($N^2 \approx 5 \cdot 10^{11}$). The error of the approximation for $M = 15$ is $q = 7.8\%$ ($q = 18.8\%$ for $M = 4$), where

$$q = \frac{\int_{SE(2)} ||f(g) - f'(g)||_2^2 \, d\mu(g)}{\int_{SE(2)} ||f(g)||_2^2 \, d\mu(g)} \quad , \tag{10}$$

where $f(g)$ in (10) is the convolved workspace performed by direct numerical integration of workspace densities and $f'(g)$ is the convolution performed by the Fourier method.

In Fig. (4b) we depict the convolved workspace density of a twelve module manipulator computed by direct integration. The density computed by the Fourier method (for $M = 4$) is depicted in Fig. (4c) (plots are depicted in the range of workspace density values from 1/10 to maximal value of density). Thus, the Fourier transform reproduces correctly the workspace density in the region of values from the maximal value of the intensity to approximately 1/10th $\sim$ 1/50th value of the maximal intensity. We note that for the twelve module manipulator the density in this region provides "almost continuous" density of states and this region is the most important for the manipulator's applications. We note that the boundary of the low intensity region may be found also by the Fourier method convolving the functions which reproduce correctly the workspace boundary of half of the manipulator, rather than the workspace density.

## 4 AN ANZATZ APPROACH TO THE DESCRIPTION OF THE WORKSPACE DENSITY AND ITS APPLICATION IN LINEAR AND NON-LINEAR INVERSE PROBLEMS

The following inverse problems arise naturally in the area of kinematic design of binary manipulators:

- Given a final desired workspace density and known parameters for the lower half of a manipulator find the kinematic parameters of the upper half of the manipulator such that the total workspace density of the manipulator fits the desired density in the best way (in the sense of the quadratic deviation). This is the so-called linear inverse problem.
- Given a final desired workspace density, find the kinematic parameters of each half of the manipulator which result in a workspace density which fits the desired density in the best way. This is a non-linear inverse problem.

The problems above may be written respectively as linear and non-linear integral convolution equations on the motion group. The linear inverse problem may be formulated as

$$(\alpha * \beta)(g) = \int_{SE(2)} \alpha(h) \, \beta(h^{-1} \circ g) \, d\mu(h) \; = \gamma(g) \quad , \tag{11}$$

where $\gamma(g)$ is the desired total workspace density, $\alpha(g)$ is the density of the lower part of manipulator and $\beta(g)$ is the unknown density of the upper part.

The non-linear inverse problem may be written as

$$(\alpha * \alpha)(g) = \int_{SE(2)} \alpha(h) \, \alpha(h^{-1} \circ g) \, d\mu(h) \; = \gamma(g) \quad , \tag{12}$$

where $\gamma(g)$ is the total desired density, and $\alpha(g)$ is the unknown density of both the lower and upper part (assuming the manipulator is homogeneous).

The direct way of solving these problems, i. e. fitting the total density to a given desired density for each value of manipulator kinematic parameters, would require multiple convolutions (one for each set of parameters) and it would be very costly computationally (since one could imagine doing hundreds of iterations before the minimum is found). The problem becomes easier if we want to fit the density of only a few modules to the given function. So we have to reduce the problem to the fitting problem for a small number of modules which may be computed by brute force. Because the total density, which describes the workspace density of whole manipulator, may be written as the convolution of the densities of manipulator segments, the problem becomes simpler in Fourier space, where the convolution is just a matrix product of Fourier transform matrices.

The first kind of problem, (11), leads to the linear matrix equation of the type

$$\hat{\beta}(p) \, \hat{\alpha}(p) = \hat{\gamma}(p) \tag{13}$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ denote the Fourier transform matrices of the lower, upper and whole desired workspace density.

For nonsingular matrix $\hat{\alpha}$ the solution is straight forward:

$$\beta(g) = \mathcal{F}^{-1}(\hat{\gamma}(p)\hat{\alpha}^{-1}(p)), \tag{14}$$


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where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

When the matrix $\hat{\alpha}$ is singular the problem may be reduced to the problem of minimization of an appropriate quadratic functional for chosen values of regularization parameters (see (Chirikjian 1996; Kyatkin and Chirikjian 1996a) for more details). For the simple functional

$$C = \int_{SE(2)} (|(\alpha * \beta)(g) - \gamma(g)|^2 +$$

$$\epsilon|\beta(g)|^2 + \nu(\beta(g), -\nabla^2_{\mathbf{a}}\beta(g))) \, d\mu(g)$$

the solution may be expressed in Fourier space as

$$\hat{\beta} = \hat{\gamma}\,\hat{\alpha}^\dagger\,(\hat{\alpha}\hat{\alpha}^\dagger + (\epsilon + \nu p^2)\mathbf{1})^{-1} \quad , \tag{15}$$

where $\mathbf{1}$ is an identity matrix, $\epsilon$ and $\nu$ are regularization parameters, and $\dagger$ denotes Hermitian conjugate. In fact, we have to choose the parameters $\epsilon$ and $\nu$ as small as possible in order to minimize the quadratic error. But the parameters cannot be taken arbitrary small because the solution starts to exhibit oscillatory and singular behavior, and the quadratic norm of the solution increases as parameters approach to zero. In fact, there is an approximate range of "boundary" values where the norm of the solution is of the order of norm of the $\alpha(g)$, for these values further reduction of the parameters must be stopped. The quadratic error, however, does not depend strongly on the exact values of parameters, so we may pick any small values of parameters in this region.

**Analytical description of workspace density with an anzatz function.**

Before we start to discuss the inverse non-linear problem we have to find an appropriate way to describe the desired workspace densities in analytical form. Observing the plots of density functions in Fig. (4b,c) we may extract the most important "global" properties of the workspace densities.

We observe that the density is "shrinking" with increasing orientation angle $\theta$. At the same time it is "rotating" in the $x-y$ plane with increasing $\theta$ and "moves" closer to the origin. We also may characterize the density by the point of maximal value of density for each fixed orientation angle. We want to incorporate these important "global" features of the workspace density into an *anzatz* [3] function which has these properties and has a relatively small number of free parameters. For "symmetric" manipulators (that is, ones with no preferred bending direction) the workspace density also has a symmetry

$$\theta \to -\theta \, , \qquad \phi \to -\phi$$

To describe symmetric workspaces for manipulators with a relatively small number of modules ($n_{mod} \lesssim 10$) we suggest to parametrize the density functional as

$$f'(r,\phi,\theta) = c_5 \, (\frac{1}{\sigma_1\sqrt{2\pi}} \, \exp(-\frac{(r-x)^2}{2\,\sigma_1^2}) \times$$

$$(1 + c_3 \, \cos\theta)^n \, \frac{1}{\sigma_2\sqrt{2\pi}} \, \exp(-\frac{(\phi - c_4\,\theta)^2}{2\,\sigma_2^2}) \tag{16}$$

where $x = 1/2\,(1+\cos\theta)\,c_1 + 1/2\,(1-\cos\theta)\,c_2$. The Gaussian term (normal distribution) centered at $x$ describes the radial dependence of the workspace density. We choose $c_2 < c_1$, so the center of the distribution moves closer to the origin with increasing in $\theta$. The term containing $c_3$ describes "shrinking" of the workspace with increasing $\theta$, the power $n$ is some positive number. We assume that for $c_3 \leq 1$ an allowed range of $\theta$ values is $|\theta| \leq \arccos(-1/c_3)$, and $f'(g) = 0$ for $\theta$ outside of this range. The term containing the $\phi$-dependence is responsible for the "rotation" of the workspace with increasing $\theta$. We illustrate the anzatz function parameters in Fig. (5).

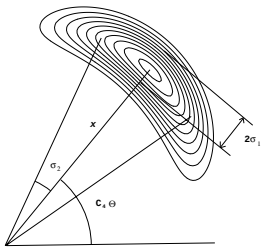


Fig. 5

We assume that $-\pi < \theta \leq \pi$, $-\pi < \phi \leq \pi$, and the density function is assumed to be $2\pi$-periodic.

We have to mention that considerable deviations from the anzatz may appear for $\theta \approx \pi$ (this is where "disconnected" regions in each $\theta$-slice of the density function may

[3] Anzatz is a term often used in the physical sciences when impirical observations are used to form a model in the absence of a well established physical principle.


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appear from counting the configurations which differ by multiples of $2\pi$ in orientation angle $\theta$, i. e. for angles which are outside the range $-\pi < \theta \leq \pi$). For a manipulator with a large number of modules ($n \gtrsim 10$) we suggest to replace the last term

$$\frac{1}{\sigma_2\sqrt{2\pi}}\,\exp(-\frac{(\phi - c_4\,\theta)^2}{2\,\sigma_2^2})$$

by the term

$$[\frac{1}{\sigma_2\sqrt{2\pi}}\,\exp(-\frac{(\phi - c_4\,\theta)^2}{2\,\sigma_2^2}) + h(\theta)\,(\frac{1}{\sigma_2\sqrt{2\pi}}\,\exp(-\frac{(\phi + c_4'\,\theta)^2}{2\,\sigma_2^2})] \tag{17}$$

where $c_4' = 1/2\,(1-\cos\theta)\,c_4 + 1/2\,(1+\cos\theta)c_6$, and $h(\theta)$ is an even function such that $h(\pi) = h(-\pi) = 1$.

We note that the $c_5$ coefficient must be determined from the condition

$$\int_{SE(2)} f(g)\,d\mu(g) \;=\; K^n$$

where $K^n$ is the total number of configurations of the manipulator. For convenience we divided the density function by $K^n$, so the function should be normalized on 1. The anzatz function describes the high-density region of workspace where approximately 90% of states are located.

As an example, for a six module manipulator with the parameters which we used before ($l_{min} = 0.12,\ l_{max} = 0.2,\ s = 0.2$) the workspace may be approximated by the anzatz function (16) with the parameters

$$c_1 = 0.71; \quad \sigma_1 = 0.09; \quad c_2 = 0.53; \quad c_3 = 1.0;$$
$$c_4 = 0.47; \quad \sigma_2 = 0.38; \quad c_5 = 8.98; \quad n = 1.1\ .$$

The error of the approximation in the sense of the quadratic norm (10) is $q = 24.0\%$.

We may describe the desired workspace using the anzatz function (16) or (17) and choosing the appropriate coefficients for this function.

**Example solution of the linear inverse problem.**

As an example, we solved the linear inverse problem for the following anzatz function (with the modified term (17)) with the coefficients

$$c_1 = 1.5; \quad c_2 = 0.9; \quad c_3 = 0.5;$$
$$c_4 = 0.5; \quad c_6 = 2.5; \quad \sigma_1 = 0.15;$$
$$\sigma_2 = 0.46 \quad n = 2.0; \quad c_5 = 4.09\ ;$$

and the choice of $h(\theta) = \exp(-(|\theta| - 2.83)^2/0.1)$, for $|\theta| \leq 2.83$; and $h(\theta) = 1$, for $|\theta| > 2.83$ (we note that the particular choice of $h(\theta)$ does not change considerably the norm of the function, because it affects only the regions of low intensity). The anzatz function with these parameters describes the desired density function $\gamma(g)$. We choose the manipulator density $\alpha_{mnp}(g)$ to be described by the workspace of the six module manipulator with the parameters

$$l_{min} = 0.12; \quad l_{max} = 0.20; \quad s = 0.20$$

The anzatz function for the given values of parameters is depicted in Fig. (6a).

We look for solutions using (15), where we put $\nu = 0$. The solution exhibits singular behavior if $\epsilon \to 0$. We choose the value of $\epsilon = 0.01$ (the norm of the solution is $||\beta||_2^2 = 30.79$). The corresponding approximate solution $\beta(g)$, found according to (15), is depicted in Fig. (7a). We note that the solution is not strictly positive. We truncated the negative part of the solution in the fitting procedure (the norm of positive part of the solution is $||\beta_{pos}||_2^2 = 26.34$).

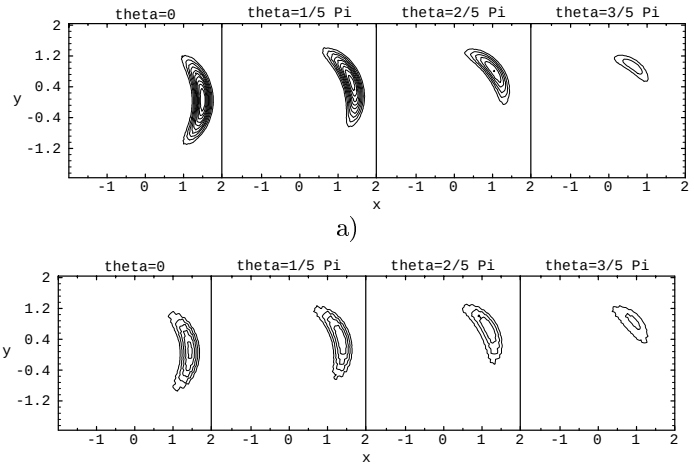


Fig. 6

Fitting of the six module manipulator density $\beta_{mnp}(g)$ to the function $\beta(g)$ by brute force in the space of three parameters $l_{min}$, $l_{max}$, $s$ (the fitting was performed at approximately 120 "points" of the parameter's space in 2 hours) gives the following values of the parameters which produce the first five smallest quadratic intermediate errors $q_{int}$ (deviation of $\beta_{mnp}(g)$ from $\beta(g)$)


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| $l_{min}$ | $l_{max}$ | $s$ | $q_{int}$ | $q$ |
|---|---|---|---|---|
| 0.12 | 0.25 | 0.19 | 55.8% | 29.4% |
| 0.12 | 0.24 | 0.19 | 55.9% | 26.1% |
| 0.12 | 0.24 | 0.20 | 56.2% | 26.8% |
| 0.12 | 0.24 | 0.21 | 56.0% | 27.6% |

(the error increases rapidly for other parameters, for $l_{min} = 0.15$; $l_{max} = 0.23$; $s = 0.19$ it is 173%). We calculated the accuracy of the approximation of the desired function $\gamma(g)$ by the final density $(\alpha_{mnp} * \beta_{mnp})(g)$ for these values of the parameters and give a value of the quadratic error $q$ in the table. We show the convolution $(\alpha_{mnp} * \beta_{mnp})(g)$ for the smallest quadratic error $q$

$$l_{min} = 0.12;\ l_{max} = 0.24;\ s = 0.19 \tag{18}$$

in Fig. (6b) (computed by the Fourier method with $M = 4$). We note that the calculated solution is an approximate one. The exact solution (value of the parameters of manipulator) is located in the vicinity of these values. If better accuracy is desired, direct fitting of $(\alpha_{mnp} * \beta_{mnp})(g)$, found by the Fourier method, may be performed for the parameters in the vicinity of (18). We found that $(\alpha_{mnp} * \beta_{mnp})(g)$ for the parameter values of $\beta_{mnp}(g)$

$$l_{min} = 0.13; \quad l_{max} = 0.22; \quad s = 0.19;$$

approximates $\gamma(g)$ with the best accuracy (in this case the error was 14.0%). The direct fitting of the convolution is, however, at least 7 times more costly computationally (for each "point" in the space of manipulator parameters) for $M = 4$, and more than 30 times for $M = 15$. Thus, the solution of the linear inverse problem (15) gives a fast way to find an approximate density function $\beta(g)$ with acceptable error.

**Example solution of the non-linear inverse problem.**

The solution of the non-linear problem may be found in a similar fashion, i. e. we find first numerically the approximate solution of the non-linear convolution equation and then find the manipulator's parameters which describe the workspace density with the smallest quadratic error.

The non-linear problem becomes a problem of a search for the "square root" of a matrix in Fourier space, i. e. the solution must satisfies the equation

$$\sum_k \hat{\alpha}(p)_{mk} \hat{\alpha}(p)_{kn} = \hat{\gamma}(p)_{mn}$$

for each value of $p$.

An algorithm for the approximate solution of the non-linear problem, which uses Schur decomposition of Fourier matrices, was described in (Kyatkin and Chirikjian 1996b). Again, the solutions depend on the regularization parameter $\epsilon$. We have to choose the value of the parameter $\epsilon$, which gives the value of the norm in the region $||f||_2^2 \lesssim 50$ (for a six module manipulator). Moreover, we have additional continuous arbitrariness of the solution related to the following fact. The square root of a Fourier matrix requires one to take the square roots of eigenvalues of the Fourier matrix $\hat{f}(p)$, the square root has two branches (positive and negative branches for real positive numbers), and we may take the different branches for different values of the parameter $p$ (see (Kyatkin and Chirikjian 1996b) for details). While it does not change the norm of the solution, it changes considerably the shape of the function. From the direct convolution of anzatz functions we may observe that the convolution of anzatz functions produces an anzatz-like function for some values of parameters. Thus, we require the "square root" solution to be anzatz-like (i.e. it may be approximated by an anzatz function with a small error). This condition may be implemented in Fourier space as follows. First, we determine approximately how to choose the branches of the square root for eigenvalues using the "trial" anzatz function, i. e. we convolve the trial function with itself and find which branch to choose comparing the square root of eigenvalues of the convolution with eigenvalues of the trial function. Then, we use this prescription for the branches of the square root to find the solution of the non-linear problem for the given desired total workspace density.

As an example we solved the non-linear problem for the following set of coefficients of anzatz function (with term (17)) which describes the desired total workspace density

$$\begin{aligned} &c_1 = 1.6; \quad c_2 = 0.95; \quad c_3 = 0.5; \\ &c_4 = 0.5; \quad c_6 = 2.5; \quad \sigma_1 = 0.15; \\ &\sigma_2 = 0.45 \quad n = 2.0; \quad c_5 = 3.85; \end{aligned} \tag{19}$$

We use a value of the regularization parameter $\epsilon = 0.03$ and determine the approximate prescription how to choose the branches of square root of eigenvalues from the trial function, which has the coefficients with half values for $c_1$, $c_2$ and $\sigma_1$ and of the same order for the other coefficients


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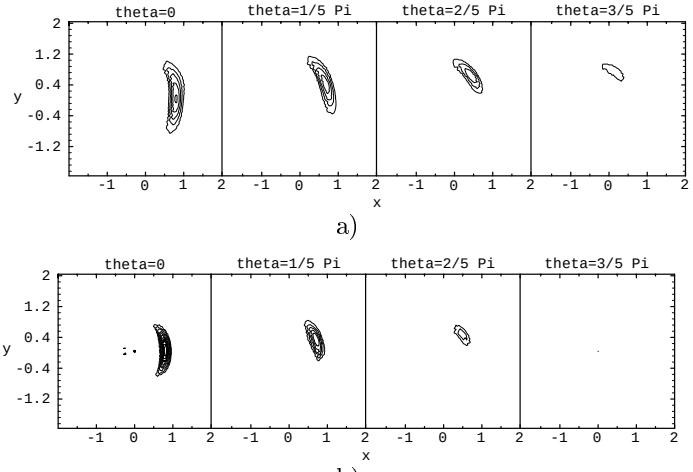
(which should give the function similar to the solution, because the value of the $c_1$ is scaled approximately by factor two for the convolution). Using this prescription we found numerically the solution of the non-linear problem which is depicted in Fig. (7b) (because the prescription is only approximate, the small deviations from anzatz-like shape appear (such as regions of the negative values), which, however, do not affect considerably the norm of the solution).




a)

b)

Fig. 7

Fitting of the six module manipulator density $\alpha_{mnp}(g)$ to the function $\alpha(g)$ performed by brute force in the space of three parameters $l_{min}$, $l_{max}$, $s$ gives the following values of the parameters which minimizes the quadratic error (first three local minima with the smallest values of the errors are shown)

$$l_{min} = 0.12; \quad l_{max} = 0.24; \quad s = 0.21; (58.1\%) \qquad (20)$$

$$l_{min} = 0.13; \quad l_{max} = 0.24; \quad s = 0.23; (59.3\%) \qquad (21)$$

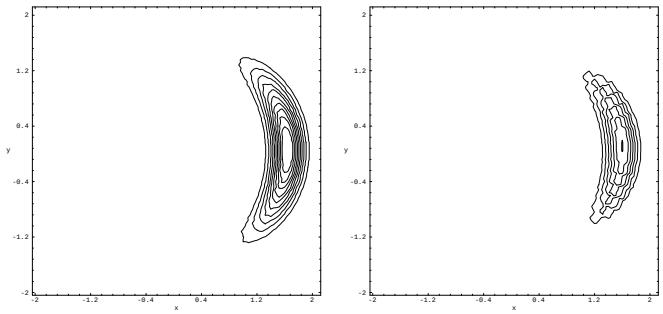
$$l_{min} = 0.13; \quad l_{max} = 0.23; \quad s = 0.21; (59.6\%) \qquad (22)$$

(the error increases rapidly for other values, for example for $l_{min} = 0.15$; $l_{max} = 0.22$; $s = 0.20$ it is 236.7%). We computed the value of the quadratic deviation of the convolved function $(\alpha_{mnp} * \alpha_{mnp})(g)$ from the desired function determined by the coefficients (19), and found that the minimum (22) has a smallest quadratic error 18.6% (the minimum (21) gives 26.8% of error, and (20) gives 47.0%). The contour plot of the desired function and the manipulator density for the parameters in (22) are shown in Fig. (8a) and (8b) for $\theta = 0$.

If better accuracy is desired the direct fitting of $(\alpha_{mnp} * \alpha_{mnp})(g)$ (computed by the Fourier convolution method) into the desired function $\gamma(g)$ may be performed for the parameter values in the vicinity of (22). We found that for the manipulators parameters

$$l_{min} = 0.14; \quad l_{max} = 0.23; \quad s = 0.24;$$

the manipulator workspace density has a smallest deviation from $\gamma(g)$ equals 14.6%. All convolutions were performed using $M = 4$.



a) b)

Fig. 8

## CONCLUSIONS

We have shown in this paper that the Fourier method is a fast method to compute convolutions of two dimensional manipulator workspace densities, which are functions on the motion group. The accuracy of the convolutions is acceptable for $M = 4$ ($9 \times 9$ Fourier matrices) and has a quadratic error in the range of $10 - 20\%$. We also suggested an *anzatz*, which allows one to describe the workspace densities of the two dimensional manipulators in a simple analytic form with few free parameters. Anzatz functions and Fourier methods on the motion group allow one to solve the linear and non-linear inverse problems of kinematic design with error around $20-30\%$. Workspace density generation, Fourier methods, and convolution methods were implemented in the C programming language. Part of the matrix computations in the inverse problems were performed using Mathematica 2.2 programs.

We note that Fourier methods for the three dimensional motion group and analytical examples of Fourier transforms and solutions of linear and non-linear inverse problems were described in (Kyatkin and Chirikjian 1996a; Kyatkin and Chirikjian 1996b), though the application of these methods to $3 - D$ manipulator design is an open and challenging numerical problem.

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