Acta Applicandae Mathematicae **53:** 89–123, 1998.

Regularized Solutions of a Nonlinear Convolution Equation on the Euclidean Group

ALEXANDER B. KYATKIN and GREGORY S. CHIRIKJIAN
Department of Mechanical Engineering, Johns Hopkins University, Baltimore, MD 21218, U.S.A.

(Received: 11 December 1996)

Abstract. In this paper we apply Fourier analysis on the two and three dimensional Euclidean motion groups to the solution of a nonlinear convolution equation. First, we review the theory of the irreducible unitary representations of the motion group and discuss the corresponding Fourier transform of functions on the motion group. The main reasons why exact solutions of this convolution equation do not exist in many cases are discussed. Techniques for regularization of the problem and numerical methods for finding approximate solutions are presented. Examples are considered and approximate solutions are found.

Mathematics Subject Classifications (1991): Primary: 43A30, 44A35; secondary: 45G10, 45L05, 45L10, 15A24, 65R20.

Key words: harmonic analysis, Fourier transform, Euclidean group, nonlinear problems, integral equations, convolution, inverse problems, regularization.

1. Introduction

Recently, the Fourier transform on the two [1] and three [2] dimensional Euclidean motion group has been applied to the solution of convolution equations on this group. Particularly, linear inverse [1, 2] and direct problems [3] have been considered. Convolution equations on the Euclidean motion group arise naturally in robotics in the area of the kinematic design of binary manipulators [4, 5].

The Euclidean motion group, SE(N)*, is the semi-direct product of $\mathbb{R}^N$ with the special orthogonal group, SO(N). That is, $\mathrm{SE}(N) = \mathbb{R}^N \rhd \mathrm{SO}(N)$. SE($N$) is an $(N+1)N/2$-dimensional Lie group. We denote elements of SE(N) as $g = (\mathbf{r}, A) \in \mathrm{SE}(N)$ where $A \in \mathrm{SO}(N)$ and $\mathbf{r} \in \mathbb{R}^N$. The group law is written as $g_1 \circ g_2 = (\mathbf{r}_1 + A_1\mathbf{r}_2, A_1A_2)$, and $g^{-1} = (-A^T\mathbf{r}, A^T)$. Alternately, one may represent any element of SE(N) as an $(N+1) \times (N+1)$ homogeneous transformation matrix of the form:

$$H(g) = \begin{pmatrix} A & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} . \tag{1}$$

* SE(N) denotes the N-dimensional Special Euclidean group.

In this paper we apply noncommutative harmonic analysis on the motion group to solve the nonlinear convolution equation

$$\int_G h(g')\, h(g'^{-1} \circ g)\, \mathrm{d}\mu(g') = (h * h)(g) = f(g)$$

for $h(g)$, where G is a motion group, $g,\ g' \in G$, $\mathrm{d}\mu(g')$ is an invariant integration measure, and $f(g)$ is any given square integrable function. We also address the solution of the more general problem of finding $h(g)$ for given $f(g)$ when

$$\underbrace{(h * \cdots * h)}_{n \text{ times}}(g) = f(g).$$

These equations arise naturally in the inverse problem of kinematic design of binary manipulators [4, 5, 3]. In this context, each function $h(g)$ represents a density of positions and orientations which one segment of a manipulator arm can reach relative to its base. The convolution generates the density of reachable frames for a concatenation of modules. The inverse problem arises when a desired total density is specified and this information must be localized to the design of each module.

First, we review briefly the matrix elements of the irreducible unitary representations of the motion group in both two and three dimensional cases and discuss the corresponding Fourier transforms of functions on the motion group.

Using techniques from noncommutative harmonic analysis, the convolution may be written as matrix multiplication, thus the convolution equation may be reduced to the problem of calculation the root of a matrix function of a single argument.

We give an example of the exact analytical solution of the convolution problem. For unitary diagonalizable Fourier transform matrices we discuss the general form of the solution. We discuss the main reasons why the exact solutions do not exist in many cases, such as the absence of exact square roots of matrices, and the appearance of singularities in the inverse Fourier transform integral.

We suggest a regularization technique for finding the approximate solutions of the problem and the corresponding numerical algorithms for the computation of the approximate solutions. An explicit example is considered where the approximate solution of the convolution equation and estimated quadratic error of this solution are found.

The paper is organized as follows. In Section 2 we list the matrix elements of the irreducible unitary representations of SE(2) and SE(3). We define the corresponding Fourier transform in Section 3. In Section 4 the convolution equation is written as a matrix equation and an analytical example is solved. General solutions for the unitary diagonalizable Fourier matrix are discussed. In Subsection 5.1 we study the case when exact solutions do not exist, and give numerical methods for calculating regularized approximate solutions in Subsection 5.2. An example of this approximation technique for solving the convolution problem is given in Subsection 5.3. The Appendix contains some useful definitions.

2. Matrix Elements of Irreducible Unitary Representations of the Motion Group

2.1. TWO DIMENSIONAL CASE

Each element of SE(2) is parametrized in either rectangular or polar coordinates as:

$$g(r_1, r_2, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & r_1 \\ \sin\theta & \cos\theta & r_2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or}$$

$$g(r, \phi, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & r\cos\phi \\ \sin\theta & \cos\theta & r\sin\phi \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $r = \mid \mathbf{r} \mid$.

A unitary representation of SE(2) (see [6, 8, 20, 21, 22] for general definition) is defined by the unitary operator:

$$\mathcal{U}(g, p)\tilde{f}(\mathbf{x}) = \mathrm{e}^{-ip(\mathbf{r}\cdot\mathbf{x})}\tilde{f}(A^T\mathbf{x}), \tag{2}$$

for each $g = (\mathbf{r}, A(\theta)) \in \mathrm{SE}(2)$. Here $p \in \mathbb{R}^+$, and $\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2$. The vector $\mathbf{x}$ is a unit vector ($\mathbf{x}\cdot\mathbf{x} = 1$), so $\tilde{f}(\mathbf{x}) = \tilde{f}(\cos\psi, \sin\psi) \equiv f(\psi)$ is a function on the unit circle. Henceforth we will not distinguish between $\tilde{f}$ and f.

Any function $f(\psi) \in \mathcal{L}^2(S^1)$ can be expressed as a weighted sum of orthonormal basis functions as $f(\psi) = \sum_n a_n \mathrm{e}^{in\psi}$. Likewise, the matrix elements of the operator $\mathcal{U}(g, p)$ are expressed in this basis as:

$$\begin{aligned} u_{mn}(g, p) &= (\mathrm{e}^{im\psi}, \mathcal{U}(g, p)\mathrm{e}^{in\psi}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-im\psi}\mathrm{e}^{-i(a_1p\cos\psi + a_2p\sin\psi)}\mathrm{e}^{in(\psi-\theta)}\,\mathrm{d}\psi \end{aligned} \tag{3}$$

$\forall\, m, n \in \mathbb{Z}$, where the inner product $(\cdot,\cdot)$ is defined as:

$$(f_1, f_2) = \frac{1}{2\pi}\int_0^{2\pi} \overline{f_1(\psi)} f_2(\psi)\,\mathrm{d}\psi.$$

It is easy to see that $(\mathcal{U}(g, p)f_1, \mathcal{U}(g, p)f_2) = (f_1, f_2)$, and that $\mathcal{U}(g, p)$ is therefore unitary with respect to this inner product.

A number of works including [20, 21, 8] have shown that the matrix elements of this representation are given by

$$u_{mn}(g(r, \phi, \theta), p) = i^{n-m}\mathrm{e}^{-i[n\theta + (m-n)\phi]}J_{n-m}(pr), \tag{4}$$

where $J_\nu(x)$ is the νth order Bessel function.

From this expression, and the fact that $\mathcal{U}(g, p)$ is a unitary representation, we have that:

$$\begin{aligned} u_{mn}(g^{-1}(r, \phi, \theta), p) &= u^{-1}_{mn}(g(r, \phi, \theta), p) \\ &= \overline{u_{nm}(g(r, \phi, \theta), p)} \\ &= i^{n-m} \mathrm{e}^{i[m\theta+(n-m)\phi]} J_{m-n}(pr). \end{aligned} \tag{5}$$

Henceforth no distinction will be made between the operator $\mathcal{U}(g, p)$ and the corresponding infinite dimensional matrix with elements $u_{mn}(g, p)$.

Symmetry property. The matrix elements are related by the symmetry

$$\overline{u_{mn}(g, p)} = (-1)^{m-n} u_{-m,-n}(g, p). \tag{6}$$

2.2. THREE DIMENSIONAL CASE

For the three dimensional motion group we parametrize the rotation matrix A in terms of the Euler angles α, β, γ of $z - x - z$ rotations, and the translation vector $\mathbf{r}$ is parametrized in terms of radial part $r = |\mathbf{r}|$ and polar and azimuthal angles θ, ϕ.

The complete orthogonal set of matrix elements of irreducible unitary representations of the three dimensional motion group may be constructed using the method of induced representations [7, 2] (for a formal definition of the induced representations see, for example, [10, 11, 9]).

Each irreducible unitary representation is indexed by an integer number $s = 0, \pm 1, \pm 2, \ldots$ and continuous Fourier parameter $p > 0$ [7, 2]. Representations, which we denote below by $\mathcal{U}^s(g, p)$, satisfy the homomorphism property

$$\mathcal{U}^s(g_1 \circ g_2, p) = \mathcal{U}^s(g_1, p) \cdot \mathcal{U}^s(g_2, p),$$

where $g \in \mathrm{SE}(3)$ and $\circ$ is the group operation.

Basis functions (see [7, 8, 2] for explicit expressions) are enumerated by integer parameters l and m (like in SO(3) representations) [12] for each s, where $l = |s|, |s|+1, |s|+2, \ldots$ and $|m| \leqslant l$. Thus, matrix elements of the irreducible unitary representations depend on integer parameters $s,\ l,\ m,\ l',\ m'$ (where $l,\ l' \geqslant |s|$ and $|m|(|m'|) \leqslant l\ (l')$) and continuous Fourier parameter $p > 0$.

To obtain the matrix elements of the unitary representations we use the group property

$$\mathcal{U}^s(\mathbf{r}, A; p) = \mathcal{U}^s(\mathbf{r}, I; p) \cdot \mathcal{U}^s(0, A; p). \tag{7}$$

The translational matrix elements are given by expression [7, 2]

$$\begin{aligned} &[l', m' \mid p, s \mid l, m](\mathbf{r}) \\ &\quad = (4\pi)^{1/2} \sum_{k=|l'-l|}^{k=l'+l} i^k \sqrt{\frac{(2l'+1)(2k+1)}{(2l+1)}}\, j_k(p\,r)\, C(k, 0; l', s \mid l, s) \times \\ &\qquad \times C(k, m-m'; l', m' \mid l, m)\, Y_k^{m-m'}(\theta, \phi), \end{aligned} \tag{8}$$

where θ, ϕ are polar and azimuthal angles of translation $\mathbf{r}$,

$$j_k(z) = \sqrt{\pi/2z}\, J_{k+1/2}(z),$$

$Y_k^m(\theta, \phi)$ are spherical functions, and $C(k, m - m'; l', m' \mid l, m)$ are Clebsch–Gordan coefficients (see, for example, [13]).

The rotation matrix elements $\mathcal{T}^l_{nm}(A)$ are

$$\mathcal{T}^l_{mn}(A) = \mathrm{e}^{-im\alpha}\,(-1)^{n-m}\,P^l_{mn}(\cos\beta)\,\mathrm{e}^{-in\gamma}, \tag{9}$$

where α, β, γ are $z{-}x{-}z$ Euler angles of the rotation. We note that the rotation matrix elements do not depend on s. Generalized Legendre polynomials $P^l_{m\,s}(\cos\theta)$ are given as in Vilenkin [6] (see appendix).

Finally, using the group property (7), the matrix elements of the unitary representation $\mathcal{U}^s(g, p)$ (for $s = 0, \pm 1, \pm 2, \ldots$) are expressed as

$$\mathcal{U}^s_{l',m';l,m}(\mathbf{r}, A; p) = \sum_{j=-l}^{j=l} [l', m' \mid p, s \mid l, j](\mathbf{r})\,\mathcal{T}^l_{jm}(A). \tag{10}$$

Because (8) contains only half-integer Bessel functions, all matrix elements may be expressed in terms of elementary functions, for example

$$\mathcal{U}^0_{0,0;0,0}(\mathbf{r}, A; p) = \frac{\sin(rp)}{rp};$$

$$\mathcal{U}^0_{1,0;0,0}(\mathbf{r}, A; p) = \frac{i\sqrt{3}\,\cos(\theta)\,\,(-\cos(rp) + \sin(rp)/(rp))}{rp}.$$

Symmetry property. We note an important symmetry property of these matrix elements [2]

$$\overline{\mathcal{U}^s_{l',m';l,m}(\mathbf{r}, A; p)} = (-1)^{(l'-l)}(-1)^{(m'-m)}\,\mathcal{U}^s_{l',-m';l,-m}(\mathbf{r}, A; p) \tag{11}$$

(see [2] for other symmetry relations of matrix elements).

3. The Fourier Transform on the Motion Group

3.1. DEFINITIONS AND PROPERTIES

Here we review the Fourier transform of functions $f(g) \in L^2(G)$, where G is a motion group. We state without proof those properties derived in [2] which have applications to the current problem.

The inner product of functions is given by

$$(f_1, f_2) = \int_G \overline{f_1(g)}\, f_2(g)\, \mathrm{d}\mu(g), \tag{12}$$

where G is SE(2) or SE(3), $g \in G$. The invariant integration measure on SE(2) is given by

$$\mathrm{d}\mu(g(r,\phi,\theta)) = \frac{1}{(2\pi)^2} r\, \mathrm{d}r\, \mathrm{d}\phi\, \mathrm{d}\theta$$

(r, ϕ are the radial and angular part of translation vector $\mathbf{r}$ and θ is the SO(2) orientation angle).

The invariant integration measure on SE(3) is

$$\mathrm{d}\mu(g) = \mathrm{d}A\, \mathrm{d}^3 r$$

where $\mathrm{d}^3 r$ is an integration measure in $\mathbb{R}^3$, given in spherical coordinates by

$$\mathrm{d}^3 r = r^2\, \mathrm{d}r \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$$

(where ϕ, θ are polar and azimuthal angle of the translation $\mathbf{r}$), and $\mathrm{d}A$ is the Haar measure on SO(3), given by

$$\mathrm{d}A = \frac{1}{8\pi^2} \sin\beta\, \mathrm{d}\beta\, \mathrm{d}\alpha\, \mathrm{d}\gamma$$

(normalized such that the volume of SO(3) is 1), α, β, γ are $z-x-z$ Euler angles.

DEFINITION. For any complex-valued function $f(g) \in L^2(G)$ on the motion group G we define the Fourier transform as

$$\mathcal{F}(f) = \hat{f}(p) = \int_G f(g)\, \mathcal{U}(g^{-1}, p)\, \mathrm{d}\mu(g),$$

where $g \in G$.

The inverse Fourier transform is defined by

$$f(g) = \mathcal{F}^{-1}(\hat{f}) = C \int_0^\infty \mathrm{Tr}(\hat{f}(p)\mathcal{U}(g,p))\, p^2\, \mathrm{d}p. \tag{13}$$

The coefficient C is 1 for SE(2) and $C = 1/(2\pi^2)$ for SE(3).

The explicit expression for the three dimensional matrix elements of the Fourier transform is given in terms of matrix elements (10) as

$$\hat{f}^s_{l',m';l,m}(p) = \int_{\mathrm{SE}(3)} f(\mathbf{r}, A)\, \overline{\mathcal{U}^s_{l,m;l',m'}(\mathbf{r}, A; p)}\, \mathrm{d}A\, \mathrm{d}^3 r, \tag{14}$$

where we have used the unitarity property of the matrix elements.

The inverse Fourier transform in the three dimensional case is

$$\begin{aligned} f(\mathbf{r}, A) \;=\; & \frac{1}{2\pi^2} \sum_{s=-\infty}^{\infty} \sum_{l'=|s|}^{\infty} \sum_{l=|s|}^{\infty} \sum_{m'=-l'}^{l'} \sum_{m=-l}^{l} \int_0^\infty p^2\, \mathrm{d}p\, \hat{f}^s_{l,m;l',m'}(p) \times \\ & \times \mathcal{U}^s_{l',m';l,m}(\mathbf{r}, A; p). \end{aligned} \tag{15}$$

The analogous expressions for the planar case can be found in [1].

Parseval/Plancherel equality. The Parseval/Plancherel equality for square-integrable functions on the motion group $G = \mathrm{SE}(n)$ $(n = 2, 3)$ is:

$$\int_G |f(g)|^2 \mathrm{d}\mu(g) = C \int_0^\infty ||\hat{f}(p)||_2^2 \, p^{n-1} \, \mathrm{d}p,$$

where $C = 1$ for SE(2) and $C = 1/(2\pi^2)$ for SE(3), and $||\hat{f}(p)||_2^2$ is the Hilbert–Schmidt norm

$$||\hat{f}||_2^2 = \mathrm{Tr}(\hat{f}\hat{f}^\dagger),$$

where $\hat{f}^\dagger$ is the Hermitian conjugate of $\hat{f}$, and Tr is the trace.

Convolution property. One of the most powerful properties of the Fourier transform of functions on $\mathbb{R}^N$ is that the Fourier transform of the convolution of two square-integrable functions

$$(f_1 * f_2)(g) = \int_G f_1(h)\, f_2(h^{-1} \circ g)\, \mathrm{d}\mu(h) \tag{16}$$

is the product of the Fourier transforms of the functions. This property persists also for the convolution of functions on the motion group, namely

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_2)\, \mathcal{F}(f_1), \tag{17}$$

only now the order of the product of Fourier transforms matters.

Symmetries. For the real function $f(g)$ we note a symmetry property of the Fourier transform, which follows from symmetry (6) and (11) of the matrix elements

$$\overline{\hat{f}_{mn}} = (-1)^{(m-n)}\, \hat{f}_{-m,-n} \tag{18}$$

(two-dimensional case), and

$$\overline{\hat{f}^s_{l',m';l,m}(p)} = (-1)^{(l'-l)}\, (-1)^{(m'-m)}\, \hat{f}^s_{l',-m';l,-m}(p) \tag{19}$$

(three-dimensional case). These symmetries are preserved under multiplication of Fourier transforms (which follows trivially from the convolution property and the fact that the convolution of real valued functions is real). These symmetries also put restrictions on the eigenvalues/vectors of the Fourier transform matrices of real-valued functions on SE(N) as follows.

THEOREM 3.1.1. *Complex eigenvalues of matrices with symmetries of the form* (18) *and* (19) *appear in conjugate pairs.*

Proof.

Case 1. SE(2).

The eigenvalue problem is written in component form as

$$\lambda x_m = \sum_n \hat{f}_{mn} x_n = (-1)^m \sum_n (-1)^n \overline{\hat{f}_{-m,-n}}\, x_n.$$

Multiplying both sides by $(-1)^m$ and regrouping terms we get

$$\sum_n (-1)^n \overline{\hat{f}_{-m,-n}} x_n = \sum_n \overline{\hat{f}_{-m,-n}}((-1)^n x_n) = \lambda((-1)^m x_m).$$

Changing the dummy variables (m, n) to $(-m, -n)$, and taking the complex conjugate of both sides, one finds:

$$\sum_n \hat{f}_{m,n}((-1)^n \overline{x_{-n}}) = \overline{\lambda}((-1)^m \overline{x_{-m}}),$$

indicating that if (λ, x_m) is an eigenvalue/vector pair, then so is $(\overline{\lambda}, (-1)^m \overline{x_{-m}})$.

Case 2. SE(3).

The eigenvalue problem is written for each s block as

$$\sum_{l,m} \hat{f}^s_{l',m';l,m}\, x^s_{lm} = \lambda\, x^s_{l',m'},$$

where there is summation over l and m.

Substitution using symmetry (19) yields

$$\sum_{l,m} (-1)^{(l'-l)} (-1)^{(m'-m)} \overline{\hat{f}^s_{l',-m';l,-m}} x^s_{lm}$$
$$= (-1)^{(l'+m')} \sum_{l,m} \overline{\hat{f}^s_{l',-m';l,-m}}((-1)^{(l+m)} x^s_{l,m}) = \lambda x^s_{l',m'}.$$

Multiplication on both sides by $(-1)^{(l'+m')}$ and complex conjugation yield:

$$\sum_{l,m} \hat{f}^s_{l',-m';l,-m}((-1)^{(l+m)} \overline{x^s_{l,m}}) = \overline{\lambda}((-1)^{(l'+m')} \overline{x^s_{l',m'}}).$$

Finally, changing the dummy variables (m, m') to $(-m, -m')$, we get

$$\sum_{l,m} \hat{f}^s_{l',m';l,m}((-1)^{(l-m)} \overline{x^s_{l,-m}}) = \overline{\lambda}((-1)^{(l'-m')} \overline{x^s_{l',-m'}}),$$

indicating that for every eigenvalue/vector pair $(\lambda, x^s_{l',m'})$, there is also a pair $(\overline{\lambda}, (-1)^{(l'-m')} \overline{x^s_{l',-m'}})$. □

Contraction of indices. It is convenient to rewrite the 4-index three-dimensional Fourier transform matrix element $\hat{f}^s_{l',m';l,m}(p)$ as a 2-index matrix $\hat{f}^s_{ij}(p)$. To satisfy the matrix product definition we arrange l, m indices in a row (we show an example for $s = 0$) 0 (for $l = 0$); $-1, 0, 1$ (for $l = 1$); $-2, -1, 0, 1, 2$ (for $l = 2$); $\ldots$, which correspond to 2-indices for $1, 2, 3, 4, \ldots$.

Explicitly

$$\hat{f}^{s}_{l',m';l,m}(p) = \hat{f}^{s}_{ij}(p), \tag{20}$$

where $i = l'(l' + 1) + m' - s^2 + 1$; $j = l(l + 1) + m - s^2 + 1$.

Thus, the three dimensional Fourier transform has a block-diagonal form, where each infinite dimensional 4-index block $\hat{f}^s$ is contracted to matrix form according to (20).

Many operational properties of the Fourier transform (i.e. transformation of differential operators on functions into algebraic operations on the Fourier transform) are derived in [2].

3.2. GENERALIZED FUNCTIONS AND THE FOURIER TRANSFORM

While the Fourier transform is generally used for harmonic analysis of square-integrable functions in this paper, there are a few notable exceptions. These are the generalized functions or distributions. These are discussed here, and used throughout the paper.

The Dirac delta function for $\mathbb{R}^N$ is defined to have the following properties:

$$\int_{\mathbb{R}^N} \delta(\mathbf{r})\mathrm{d}^N r = 1, \qquad \int_{\mathbb{R}^N} f(\mathbf{r})\delta(\mathbf{x} - \mathbf{r})\,\mathrm{d}^N r = (f * \delta)(\mathbf{x}) = f(\mathbf{x}),$$

where $\mathrm{d}^N r = \mathrm{d}r_1\mathrm{d}r_2 \cdots \mathrm{d}r_N$.

The Dirac delta function on $\mathrm{SO}(N)$ has the analogous properties:

$$\int_{\mathrm{SO}(N)} \delta(\mathcal{R})\,\mathrm{d}\mathcal{R} = 1, \qquad \int_{\mathrm{SO}(N)} f(\mathcal{R})\delta(\mathcal{R}^T R)\,\mathrm{d}\mathcal{R} = (f * \delta)(R) = f(R).$$

It follows directly from the invariance of integration under shifts and inversion of the arguments of functions on $\mathbb{R}^N$ and $\mathrm{SO}(N)$ that

$$\delta(\mathbf{x} - \mathbf{r}) = \delta(\mathbf{r} - \mathbf{x}) \quad \text{and} \quad \delta(\mathcal{R}^T R) = \delta(R^T \mathcal{R}).$$

The delta function for $\mathrm{SE}(N)$ is the product of these:

$$\delta(g) = \delta(\mathbf{r})\delta(A) \quad \text{for } g = (\mathbf{r}, A) \in \mathrm{SE}(N).$$

The integration over $\mathrm{SE}(N)$ in the convolution integral (16) may be rewritten as integration over position and orientation separately:

$$(f_1 * f_2)(\mathbf{x}, R) = \int_{\mathrm{SO}(N)} \int_{\mathbb{R}^N} \rho_1(\xi, \mathcal{R})\, \rho_2(\mathcal{R}^T(\mathbf{x} - \xi), \mathcal{R}^T R)\,\mathrm{d}\xi\,\mathrm{d}\mathcal{R}, \tag{21}$$

where $g = (\mathbf{x}, R)$ and $h = (\xi, \mathcal{R})$ are elements of $\mathrm{SE}(N)$.

Using this notation, it is easy to see that the Fourier transforms of functions such as $f(R)\delta(\mathbf{x})$ and $f(\mathbf{x})\delta(R)$ reduce to

$$\int_{\mathrm{SO}(N)} f(R)\,\mathcal{U}(\mathbf{0}, R^T; p)\mathrm{d}R$$

and

$$\int_{\mathbb{R}^N} f(\mathbf{x})\,\mathcal{U}(-\mathbf{x}, I; p)\mathrm{d}^N x,$$

respectively.

3.3. EXAMPLES OF THE FOURIER TRANSFORM

Let us calculate the Fourier transform of the following function on the three dimensional motion group

$$f(\mathbf{r}, A) = F(r)\ \cos\theta\ \cos\beta, \tag{22}$$

where θ is the polar angle of $\mathbf{r}$ and β is a Euler angle (around the x-axis) of rotation A. Because $\mathcal{U}^1_{00}(A) = \cos\beta$, it may be shown from (8) and (10) that only $\hat{f}^0_{1,0;0,0}(p)$; $\hat{f}^0_{1,0;1,0}(p)$; $\hat{f}^0_{1,0;2,0}(p)$; $\hat{f}^1_{1,0;1,0}(p)$; $\hat{f}^1_{1,0;2,0}(p)$; $\hat{f}^{-1}_{1,0;1,0}(p)$; $\hat{f}^{-1}_{1,0;2,0}(p)$ may be nonzero. Direct computations for the case when $F(r) = \mathrm{e}^{-r}$ show that the Fourier transform elements are (we show only the nonzero matrix elements)

$$\begin{aligned}
\hat{f}^0_{1,0;0,0}(p) &= -\frac{8i\pi}{3\sqrt{3}}\,\frac{p}{(1+p^2)^2};\\
\hat{f}^0_{1,0;2,0}(p) &= -\frac{16i\pi}{\sqrt{135}}\,\frac{p}{(1+p^2)^2};\\
\hat{f}^1_{1,0;2,0}(p) &= -\frac{8i\pi}{3\sqrt{5}}\,\frac{p}{(1+p^2)^2};\\
\hat{f}^{-1}_{1,0;2,0}(p) &= -\frac{8i\pi}{3\sqrt{5}}\,\frac{p}{(1+p^2)^2}.
\end{aligned} \tag{23}$$

For the inverse Fourier transform we obtain the following expression for the trace in Equation (13)

$$\mathrm{Tr}(\hat{f}(p)\mathcal{U}(g, p)) = 8\pi\cos\theta\ \cos\beta\,\frac{(\sin(pr) - pr\ \cos(pr))}{(1+p^2)^2\,pr^2}.$$

The p integration in (13) reproduces the original function.

If instead we choose $F(r) = r^4\mathrm{e}^{-r}$, then the nonzero matrix elements are

$$\hat{f}^0_{1,0;0,0}(p) = -\frac{64i\pi}{\sqrt{3}}\,\frac{p\,(35 - 42p^2 + 3p^4)}{(1+p^2)^6};$$

$$\hat{f}^{0}_{1,0;2,0}(p) = -\frac{128i\pi}{\sqrt{15}}\,\frac{p\,(35-42p^2+3p^4)}{(1+p^2)^6};$$
$$\hat{f}^{1}_{1,0;2,0}(p) = -\frac{64i\pi}{\sqrt{5}}\,\frac{p\,(35-42p^2+3p^4)}{(1+p^2)^6};$$
$$\hat{f}^{-1}_{1,0;2,0}(p) = -\frac{64i\pi}{\sqrt{5}}\,\frac{p\,(35-42p^2+3p^4)}{(1+p^2)^6}. \tag{24}$$

The inverse Fourier transform again gives the original function.

After contraction of four indices to two using (20), the Fourier transform of the examples (24) and (23) may be written as a block-diagonal matrix

$$\hat{F} = \begin{bmatrix} \boxed{\hat{F}_{-1}} & & \\ & \boxed{\hat{F}_0} & \\ & & \boxed{\hat{F}_1} \end{bmatrix} \tag{25}$$

where we truncate all zero elements for $l(l') > 2$.

The nonzero blocks are the 9×9 matrix $\hat{F}_0$ and two 8×8 matrices $\hat{F}_{-1}$, $\hat{F}_1$ (lower indices correspond to s index). Using (20) these matrices may be depicted as

$$\hat{F}_0 = \begin{bmatrix} 0 & \dots & 0 & & 0 \\ \vdots & & & & \vdots \\ \hat{f}^0_{31} & \dots & \hat{f}^0_{37} & & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & 0 & & 0 \end{bmatrix}, \tag{26}$$

where $\hat{f}^0_{31} = \hat{f}^0_{1,0;0,0}(p)$, $\hat{f}^0_{37} = \hat{f}^0_{1,0;2,0}(p)$. The other matrices are

$$\hat{F}_{\pm 1} = \begin{bmatrix} 0 & \dots & 0 & & 0 \\ 0 & \dots & \hat{f}^{\pm 1}_{26} & & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & 0 & & 0 \end{bmatrix}, \tag{27}$$

where $\hat{f}^{\pm 1}_{26} = \hat{f}^{\pm 1}_{1,0;2,0}(p)$.

4. Application of the Fourier Transform to Solution of the Nonlinear Inverse Convolution Equation

In this section we address the solution of the convolution equation

$$(h * h)(g) = f(g), \tag{28}$$

where $g \in \mathrm{SE}(3)$ or SE(2), and $f(g)$ is not only square integrable, but also rapidly decreasing. ⋆ Unless otherwise specified, these conditions on $f(g)$ will be assumed throughout the paper.

We want to find $h(g)$ in (28) for given $f(g)$. Solution techniques for this equation generalize naturally to

$$\underbrace{(h * \cdots * h)}_{n \text{ times}}(g) = f(g).$$

This is addressed in the next section.

After taking the Fourier transform, Equation (28) may be written in the form

$$\hat{H}(p) \cdot \hat{H}(p) = \hat{F}(p). \tag{29}$$

The original problem then becomes a problem of finding the square root of the Fourier transform operator. We truncate the Fourier transform matrix and reduce this problem to a finite matrix equation. In the two dimensional case this equation is a matrix equation with square matrices of dimension $2M + 1$ if we truncate the Fourier transform matrix at $|m, n| = M$. In the three dimensional case this equation is, in fact, $2L + 1$ matrix equations (one for each square block enumerated by s, $|s| \leqslant l, l')$ of dimension $(L + 1)^2 - s^2$ if we truncate the transform at $|l, l'| = L$ for each s.

We end the introduction of this section with the observation that if $h(g)$ is a solution of $(h * h)(g) = f(g)$ then $h^{'}(g)$ is a solution of $(h^{'} * h^{'})(g) = f^{'}(g)$ where $h^{'}(g) = h(g_1^{-1} \circ g \circ g_1)$, $f^{'}(g) = f(g_1^{-1} \circ g \circ g_1)$, and $g_1 \in G$ is arbitrary. This follows directly from the definition of convolution and the invariance of integration of functions under shifts and inversion of the argument.

In Subsection 4.1 general classes of functions for which this nonlinear inverse problem can be solved exactly are discussed. In Subsection 4.2 an explicit example which does not fall into these special categories is illustrated.

4.1. EXACT SOLUTIONS FOR UNITARY DIAGONALIZABLE FOURIER TRANSFORMS

Here we address the problem of solving the convolution Equation (28), when the Fourier transform of the function $f(g)$ may be brought to diagonal form by unitary transformation:

$$\hat{F}(p) = U(p)\, \hat{F}_{\text{diag}}(p)\, U^{\dagger}(p). \tag{30}$$

For these functions we may always find exact complex solutions if the function is band-limited, (i.e. a finite Fourier matrix is an exact Fourier transform). If the

⋆ We define a rapidly decreasing function on SE(N) as one for which $\int_{\mathrm{SE}(N)} |\mathbf{r}|^n |f(g)|\, \mathrm{d}\mu(g)$ and $\int_0^{\infty} |\hat{F}(p)| p^{n+N-1} \mathrm{d}p$ are finite for all finite powers of n.

function $f(g)$ is not band-limited then we may find band-limited approximations (i.e. with truncated Fourier transform) which become more accurate when higher harmonics are retained.

We have to note that the condition in (30) is true only for a very restrictive class of functions. In general, a square Fourier matrix (truncated) may only be brought by unitary transformation to upper triangular form. We discuss questions of existence of solutions and general algorithms for search of solutions in the next section.

It is well known that a matrix $\hat{F}(p)$ may be diagonalized by the unitary matrix U, if and only if it is a normal matrix [19], i.e.

$$\hat{F}(p)\hat{F}^{\dagger}(p) = \hat{F}^{\dagger}(p)\hat{F}(p). \tag{31}$$

This leads to the question of what kind of functions have Fourier transforms with this property. Thus the following proposition:

PROPOSITION 4.1.1. *Functions $f(g) \in L^2(G)$ with Fourier transforms satisfying* (31) *include:*

(a) $f(g)$ *which satisfing the condition*
$$\overline{f(g)} = \pm f(g^{-1})$$
We call these functions symmetric (*antisymmetric*).

(b) $f(g)$ *which is a class function, i.e.*
$$f(g) = f(h^{-1} \circ g \circ h)$$
for any $g, h \in G$.

(c) $f(g)$ *with a Fourier transform matrix which is proportional to a unitary matrix.*

Proof. Cases (a) and (b) may be shown easily using the Fourier transform definition, the unitarity of $\mathcal{U}(g, p)$, and the invariance of integration with $\mathrm{d}\mu(g)$ under shifts and inversion of the argument. In particular, it is easy to see that the Fourier transform of a symmetric (antisymmetric) function is a Hermitian (skew-Hermitian) matrix

$$\hat{F}^{\dagger}(p) = \pm\hat{F}(p).$$

Case (c) follows directly from the definition of unitary matrices. □

We may also find normal matrices which are not in these categories. For example, a function with the Fourier transform matrix $\hat{f}^{\dagger}(p) = \mathrm{e}^{ic(p)}\hat{f}(p)$ (where $c(p)$ is some function) is also normal. The next two proofs examine how broad the sets of class and symmetric/antisymmetric functions are for SE(N).

LEMMA 4.1.2. *There are no class functions in L^2(SE(N)) for which $\int_{\mathrm{SE}(N)} |f(g)|\,\mathrm{d}\mu(g) > 0$.*

Proof. For a function to be a class function on SE(N) necessary conditions are that $f(g_1^{-1} \circ g \circ g_1) = f(g)$ for $g_1 = (\mathbf{0}, A_1)$ and $f(g_2^{-1} \circ g \circ g_2) = f(g)$ for $g_2 = (\mathbf{r}_1, I)$. That is, the definition must hold for general automorphisms, and so it must hold for pure rotations and translations individually. Using the notation $f(g) = f(\mathbf{r}, A)$, these conditions are written as:

$$f(\mathbf{r}, A) = f(A_1^T \mathbf{r}, A_1^T A A_1) \tag{32}$$

and

$$f(\mathbf{r}, A) = f(\mathbf{r} + (A - 1)\mathbf{r}_1, A). \tag{33}$$

Equation (33) can only be true for arbitrary $A \neq I$ if it has no dependence on $\mathbf{r}$. This leads to:

Case 1. $f(\mathbf{r}, A) = f_1(A)$.

If, however, $f(\mathbf{r}, A) = 0$ for all $A \neq I$ then Equation (33) is also satisfied. The only way this can be true and $\int_{\mathrm{SE}(N)} |f(g)| \,\mathrm{d}\mu(g) > 0$ is when:

Case 2. $f(\mathbf{r}, A) = f_2(\mathbf{r})\delta(A)$.

Neither of these functions are square integrable on SE(N). □

It is interesting to note that in order to satisfy (32), $f_1(A)$ must be a class function on SO(N), and $f_2(\mathbf{r}) = f_2(|\mathbf{r}|) = f_2(r)$. The class functions for SO(3) are functions of the matrix invariants. That is, $f_1(A) = f_1(I_1(A), I_2(A), I_3(A))$, where $\det(\lambda 1 - A) = \lambda^3 - I_1(A)\lambda^2 + I_2(A)\lambda - I_3(A) = 0$. Note that $I_3(A) = \det(A) = +1$, and any element of SO(3) can be written as $A(\theta, \omega) = RZ(\theta)R^T$ where θ and ω are the angle and axis of rotation, $Z(\theta)$ is rotation around the z-axis, and R is any rotation matrix with third column ω . * From this, it is clear that $I_1(A)$ and $I_2(A)$ can only be functions of the rotation angle, and so $f_1(A) = f_1(\theta)$.

THEOREM 4.1.3. *There exist nontrivial square-integrable symmetric and anti-symmetric functions on* SE(N).

Proof. The proof is by construction. Let f_i: $\mathbb{R}^{N\times N} \times \mathbb{R}^N \to \mathbf{C}$ (or $\mathbb{R}$) for $i = 1, 2, 3, 4$ be square integrable functions on $\mathbb{R}^{N\times N\times N}$ such that

$$f_1((-1)^n B, (-1)^m \mathbf{y}) = f_1(B, \mathbf{y}),$$

$$f_2((-1)^n B, (-1)^m \mathbf{y}) = (-1)^m f_2(B, \mathbf{y}),$$

$$f_3((-1)^n B, (-1)^m \mathbf{y}) = (-1)^n f_3(B, \mathbf{y}),$$

$$f_4((-1)^n B, (-1)^m \mathbf{y}) = (-1)^{(n+m)} f_4(B, \mathbf{y}),$$

* Any element of SO(3) can be parametrized as $A(\theta, \omega) = \mathrm{e}^{\theta[\omega]} = I + \sin\theta[\omega] + (1 - \cos\theta)[\omega]^2$ where θ is the angle of rotation, and ω is the unit vector specifying the axis of rotation. The matrix $[\omega]$ is the skew-symmetric matrix such that $[\omega]\mathbf{x} = \omega \times \mathbf{x}$ for any vector $\mathbf{x} \in \mathbb{R}^3$.

for all $B \in \mathbb{R}^{N\times N}$ and $\mathbf{y} \in \mathbb{R}^N$ and $m, n \in \{0, 1\}$. Then it is easy to confirm by direct substitution of $g^{-1} = (-A^T\mathbf{r}, A^T)$ for $g = (\mathbf{r}, A) \in \mathrm{SE}(N)$ in the above functions that

$$f_1(A + A^T, A^{-1/2}\mathbf{r}),$$
$$f_3(A + A^T, A^{-1/2}\mathbf{r}),$$
$$f_1(A - A^T, A^{-1/2}\mathbf{r}),$$
$$f_4(A - A^T, A^{-1/2}\mathbf{r})$$

are symmetric, and

$$f_2(A + A^T, A^{-1/2}\mathbf{r}),$$
$$f_4(A + A^T, A^{-1/2}\mathbf{r}),$$
$$f_3(A - A^T, A^{-1/2}\mathbf{r}),$$
$$f_2(A - A^T, A^{-1/2}\mathbf{r})$$

are antisymmetric, and they are all square integrable by definition (the $A^{-1/2}$ is defined as rotation around the same axis as in A by half of the angle in opposite direction). For instance if $f(g) = f(\mathbf{r}, A) = f_1(A + A^T, A^{-1/2}\mathbf{r})$, then $f(g^{-1}) = f(-A^T\mathbf{r}, A^T) = f_1(A^T + A, (A^T)^{-1/2}(-A^T\mathbf{r}))$. But since any matrix commutes with powers of itself, and $A^T = A^{-1}$, $(A^T)^{-1/2}(-A^T\mathbf{r}) = (A^{-1})^{-1/2}(-A^{-1}\mathbf{r}) = (A^{1/2}(-A^{-1}\mathbf{r}) = -A^{-1/2}\mathbf{r}$. Clearly then, $f(g^{-1}) = f_1(A^T + A, -A^{-1/2}\mathbf{r}) = f(g)$ in this case. The other cases follow in the same way. □

In the case of SE(3), if the rotation matrix is parametrized using the axis and angle of rotation, $A = A(\theta, \omega)$, and the direction of the axis of rotation is parametrized by polar and azimuthal angles, $\omega = \omega(\alpha_1, \alpha_2)$, then functions of the form $f(\theta, \alpha_1, \alpha_2; r)$ where $f(\pm\theta, \alpha_1, \alpha_2; r) = \pm f(\theta, \alpha_1, \alpha_2; r)$ are symmetric/anti-symmetric.

Solving $h * h = f$ When $\hat{f}$ is Normal

The solution of the convolution Equation (29) is given by

$$\hat{H}(p) = U(p)\, X(p)\, \hat{F}_{\mathrm{diag}}^{1/2}(p)\, X^{-1}(p)\, U^{\dagger}(p), \tag{34}$$

where $U(p)$ is a unitary transformation, $\hat{F}_{\mathrm{diag}}^{1/2}(p)$ is a square root of the diagonalized Fourier transform $\hat{F}_{\mathrm{diag}}(p)$, and $X(p)$ is a nonsingular matrix which commutes with $\hat{F}_{\mathrm{diag}}(p)$ (i.e. $X(p)\, \hat{F}_{\mathrm{diag}}(p) = \hat{F}_{\mathrm{diag}}(p)\, X(p)$, but it may not commute with $\hat{F}_{\mathrm{diag}}^{1/2}(p)$). For construction of X see [15]. In our case it is a matrix where diagonal elements are arbitrary nonzero parameters (functions of p) and the elements $X_{ij}(p)$ $(i \neq j)$ are zero if $(\hat{F}_{\mathrm{diag}}(p))_{ii} \neq (\hat{F}_{\mathrm{diag}}(p))_{jj}$. Otherwise $X_{ij}(p)$ is an arbitrary function (under the condition that the inverse Fourier transform integral exists). Using the QR decomposition [19, 15], the X matrix may be written as

$$X = \tilde{U}\tilde{X}, \tag{35}$$

where $\tilde{X}$ is an upper triangular matrix, and $\tilde{U}$ is a unitary matrix. Then it is clear that $\tilde{X}(p)\,\hat{F}^{1/2}_{\mathrm{diag}}(p)\,\tilde{X}^{-1}(p)$ is an upper triangular matrix, which has the same diagonal elements as $\hat{F}^{1/2}_{\mathrm{diag}}(p)$ matrix.

We note that the solution (34) with $X(p)$ chosen as in (35) with $\tilde{X} = X_{\mathrm{diag}}$ (in particular, when $X(p)$ is a unit matrix) is a 'minimal' solution, in the sense that it minimizes the norm $\int_G |h(g)|^2\,\mathrm{d}\mu(g)$. This is clear from the fact that

$$\int_G |h(g)|^2\,\mathrm{d}\mu(g) = C\int_0^\infty ||\hat{\tilde{H}}(p)||_2^2\, p^{n-1}\mathrm{d}p,$$

where $n = 2$, $C = 1$ for SE(2), and $n = 3$, $C = 1/(2\pi^2)$ for SE(3). Here

$$\hat{\tilde{H}}(p) = \tilde{X}(p)\,\hat{F}^{1/2}_{\mathrm{diag}}(p)\,\tilde{X}^{-1}(p).$$

The $\hat{\tilde{H}}(p)$ gives the same contributions from the diagonal elements for both diagonal and nondiagonal $\tilde{X}(p)$, but has additional nonnegative contributions from the nondiagonal elements for nondiagonal $\tilde{X}(p)$.

The solution, in general, is not unique, because the square root has two branches, and the solution may depend on arbitrary continuous functions, which come from the matrix $X(p)$.

While we may always find complex solutions for unitary diagonalizable function $f(g)$, real solutions $h(g)$ do not always exist for given real function $f(g)$ with unitary diagonalizable Fourier transform.

Consider, for example, a band-limited function on SE(3) with the Fourier transform matrix

$$\hat{F}^0(p) = -f_0(p)\cdot\mathbf{1},$$

where $\mathbf{1}$ is 4×4 unit matrix, and $f_0(p)$ is a strictly positive function. This Fourier transform gives a real function $f(g)$, because symmetry relations (19) are satisfied. However, the square root of this matrix, which gives a real function $h(g)$, does not exist. This is clear, because the $\hat{F}^{1/2}(p)_{11}$ element is always imaginary, which violates symmetry (19). So it gives an imaginary contribution to the Trace in the inverse Fourier transform (13), because the $\mathcal{U}^0_{0,0;0,0}(g,p)$ element is real and this contribution cannot be cancelled for arbitrary $f_0(p)$.

However, we may find criteria when the real solutions of functions with unitary diagonalizable Fourier transform (with an additional condition that the unitary transformation matrix also satisfies the symmetry (18) or (19)) may be easily found. In particular, we have:

THEOREM 4.1.4. *A solution $h(g)$ in $L^2(\mathrm{SE}(N),\mathbb{R})$ (for $N = 2, 3$) of Equation* (29) *may always be found for a rapidly decreasing real function $f(g)$ with* *a unitary diagonalizable Fourier transform matrix* (*and the transformation matrix* $U(p)$ *satisfying symmetry* (18) *or* (19)), *if the elements* $(\hat{F}_{\text{diag}})^{s}_{l,0;l,0}(p)$ (*for* $s = 0, \pm 1, \pm 2, \ldots$ *and* $l = |s|, |s| + 1, \ldots$) *in the Fourier transform* (*for the three dimensional case*) *or the element* $\hat{F}_{00}(p)$ (*for the two-dimensional case*) *are nonnegative functions.*

Proof. We prove here the theorem for the three-dimensional case. The proof for the two dimensional case is analogous.

The Fourier transform matrix may be written as

$$\begin{aligned}\hat{F}_{\text{diag}}(p) &= U^{\dagger}(p)\, \hat{F}(p)\, U(p) \\ &= \int_{\mathrm{SE}(3)} f(g)\, U^{\dagger}(p)\, \mathcal{U}^{\dagger}(g, p)\, U(p)\, \mathrm{d}\mu(g).\end{aligned}$$

The matrix elements of $U^{\dagger}(p)\, \mathcal{U}^{\dagger}(g, p)\, U(p)$ still have the same symmetry property as given in (11), because it is a product of matrices with this symmetry [2]. Therefore, for the real-valued function $f(g)$ we have the symmetry of the eigenvalues

$$\overline{(\hat{F}_{\text{diag}})^{s}_{l,m;l,m}(p)} = (\hat{F}_{\text{diag}})^{s}_{l,-m;l,-m}(p). \tag{36}$$

We always may take a branch of the square root which satisfies (36)

$$\overline{(\hat{F}^{1/2}_{\text{diag}})^{s}_{l,m;l,m}(p)} = (\hat{F}^{1/2}_{\text{diag}})^{s}_{l,-m;l,-m}(p) \tag{37}$$

for each $m \neq 0$. Moreover, the $(\hat{F}^{1/2}_{\text{diag}})^{s}_{l,0;l,0}(p)$ elements are real, because elements $(\hat{F}_{\text{diag}})^{s}_{l,0;l,0}(p)$ are nonnegative.

Then, the inverse Fourier transform of $h(p) = U(p)\, \hat{F}^{1/2}_{\text{diag}}(p)\, U^{\dagger}(p)$ (i.e. we chose the $X(p)$ matrix to be a unit matrix) gives a real solution

$$\begin{aligned}h(g) &= \frac{1}{2\pi^2} \int_0^{\infty} \mathrm{Tr}(U(p)\, \hat{F}^{1/2}_{\text{diag}}(p)\, U^{\dagger}(p)\, \mathcal{U}(g, p))\, p^2\, \mathrm{d}p \\ &= \frac{1}{2\pi^2} \int_0^{\infty} \mathrm{Tr}(\, \hat{F}^{1/2}_{\text{diag}}(p) \cdot U^{\dagger}(p)\, \mathcal{U}(g, p)\, U(p))\, p^2\, \mathrm{d}p. \end{aligned} \tag{38}$$

The matrix elements of $\hat{F}^{1/2}_{\text{diag}}(p)$ and $U^{\dagger}(p)\, \mathcal{U}(g, p)\, U(p)$ satisfy (19) and (11). The solution (38) is therefore real, because complex conjugate elements $(l, m; l, m)$ and $(l, -m; l, -m)$ always come in pairs (for $m \neq 0$) in the Trace in Equation (38) and the $(l, 0; l, 0)$ elements are real. Hence, we have proven the existence of real solutions provided the nonzero elements of $\hat{F}^{1/2}_{\text{diag}}$ decay rapidly enough to zero as $p \to \infty$ for the inverse Fourier transform to converge. This is guaranteed when $f(g)$ is rapidly decreasing. □

4.2. ANALYTICAL EXAMPLE OF EXACT SOLUTION OF THE CONVOLUTION EQUATION

As an illustration, we solve here the three-dimensional convolution Equation (29) for a simple band-limited function (i.e. a function for which only a finite number of harmonics contribute to the Fourier transform) given in (22). For this function exact solutions (we find particular real solutions) may be easily found. We note that in many practical cases solutions (even complex) do not exist. We discuss the methods for regularization of the problem and numerical methods for finding the approximate solutions in the next section.

For the function (22) the Fourier transform is given in (24), and it is written as a finite matrix (with three nonzero blocks) in (26) and (27). For this simple matrix the convolution Equation (29) may be solved directly, i.e. the solution $\hat{H}_{ij}$ may be found from $\sum_k \hat{H}_{ik}\hat{H}_{kj} = \hat{F}_{ij}$. We discuss more general algorithms for finding the square root (and any nth-root) of a matrix in the next section. For general methods for finding the square root (and for criteria for the existence of the solutions), see also [15].

It may be checked that the 9×9 matrix

$$\hat{H}_0 = \begin{bmatrix} 0 & 0 & \dots & 0 & \dots & & & \dots & 0 \\ \hat{h}^0_{21} & 0 & & & & & & & \vdots \\ 0 & \hat{h}^0_{32} & 0 & \hat{h}^0_{34} & 0 & \hat{h}^0_{36} & 0 & \hat{h}^0_{38} & 0 \\ \hat{h}^0_{41} & 0 & \dots & 0 & \dots & & & \dots & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & \dots & 0 & \dots & & \hat{h}^0_{67} & 0 & 0 \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & \dots & 0 & \dots & & \hat{h}^0_{87} & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & & & \dots & 0 \end{bmatrix} \tag{39}$$

is a solution of the equation $\hat{H}_0^2 = \hat{F}_0$ (for the $s = 0$ block), where $\hat{F}_0$ is given in (26), for the matrix elements

$$\begin{aligned} \hat{h}^0_{21} = \hat{h}^0_{41} &= -\frac{\sqrt{32\,p\,\pi}\,(35 - 42p^2 + 3p^4)}{3^{1/4}\,(1+p^2)^4}; \\ \hat{h}^0_{32} = \hat{h}^0_{34} &= \frac{i\sqrt{32\,p\,\pi}}{3^{1/4}\,(1+p^2)^2}; \\ \hat{h}^0_{36} = \hat{h}^0_{38} &= -\frac{8\sqrt{p\pi}\,(35 - 42p^2 + 3p^4)}{15^{1/4}\,(1+p^2)^4}; \\ \hat{h}^0_{67} = \hat{h}^0_{87} &= \frac{8\,i\sqrt{p\pi}}{15^{1/4}\,(1+p^2)^2}. \end{aligned} \tag{40}$$

Similarly, a square root of matrix (27) $\hat{F}_{\pm 1}$ may be chosen in the form of the 8×8 matrix

$$\hat{H}_{\pm 1} = \begin{bmatrix} 0 & 0 & 0 & \dots & & \hat{h}_{16}^{\pm 1} & 0 & 0 \\ \hat{h}_{21}^{\pm 1} & 0 & \hat{h}_{23}^{\pm 1} & \dots & & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \hat{h}_{36}^{\pm 1} & 0 & 0 \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & \dots & & 0 & 0 & 0 \end{bmatrix}, \tag{41}$$

where matrix elements may be taken to be

$$\begin{aligned} \hat{h}_{21}^{\pm 1} &= \hat{h}_{23}^{\pm 1} = \frac{i\sqrt{32\, p\, \pi}}{5^{1/4}\,(1+p^2)^2}; \\ \hat{h}_{16}^{\pm 1} &= \hat{h}_{36}^{\pm 1} = -\frac{\sqrt{32\, p\, \pi}\,(35 - 42p^2 + 3p^4)}{5^{1/4}\,(1+p^2)^4}. \end{aligned} \tag{42}$$

It may be shown, using the symmetry properties (11) and the contraction of indices (20), that the choice of matrix elements in the form (40) and (42) gives a real-valued function $h(g)$ for all $g \in \mathrm{SE}(3)$. In Figure 1a we depicted the $\theta - \beta$ dependence of the original function $f(g)$ (22) for $r = 1$; $\phi = \pi/2$; $\alpha = 0$; $\gamma = 0$. In Figure 2a the $\theta - \beta$ dependence of the solution $h(g)$ is depicted for the same parameters.

We note here that the solution (which gives real $h(g)$) of the square root Equation (29) is not unique. In fact, the matrix elements

$$k(p)\hat{h}_{21}^{0}(p);\quad k(p)\hat{h}_{41}^{0}(p);\quad \frac{1}{k(p)}\hat{h}_{32}^{0}(p);\quad \frac{1}{k(p)}\hat{h}_{34}^{0}(p) \tag{43}$$

(and the analogous transformations for other elements) still give the solution of the matrix Equation (29) ($k(p)$ can be any real function for which the inverse Fourier transform exists). We note that $k(p) = 1$ is not the optimal solution in the sense that for real valued $k(p)$ the part of the integral $\int_0^\infty ||\hat{h}(p)||_2^2 p^2\,\mathrm{d}p$ which depends on the four terms above is of the form

$$\int_0^\infty \left[k^2(p)(|\hat{h}_{21}^{0}(p)|^2 + |\hat{h}_{41}^{0}(p)|^2) + \frac{1}{k(p)}(|\hat{h}_{32}^{0}(p)|^2 + |\hat{h}_{34}^{0}(p)|^2)\right] p^2\,\mathrm{d}p.$$

Minimizing under the integral sign with respect to $k(p)$ indicates that the optimal value is

$$k(p) = \left(\frac{|\hat{h}_{32}^{0}(p)|^2 + |\hat{h}_{34}^{0}(p)|^2}{|\hat{h}_{21}^{0}(p)|^2 + |\hat{h}_{41}^{0}(p)|^2}\right)^{1/4}.$$

We note that $k(p)$ has singularities at $p = (7 \pm 4\sqrt{7/3})^{1/2}$. These singularities, however, are integrable and the inverse Fourier integral is well defined.

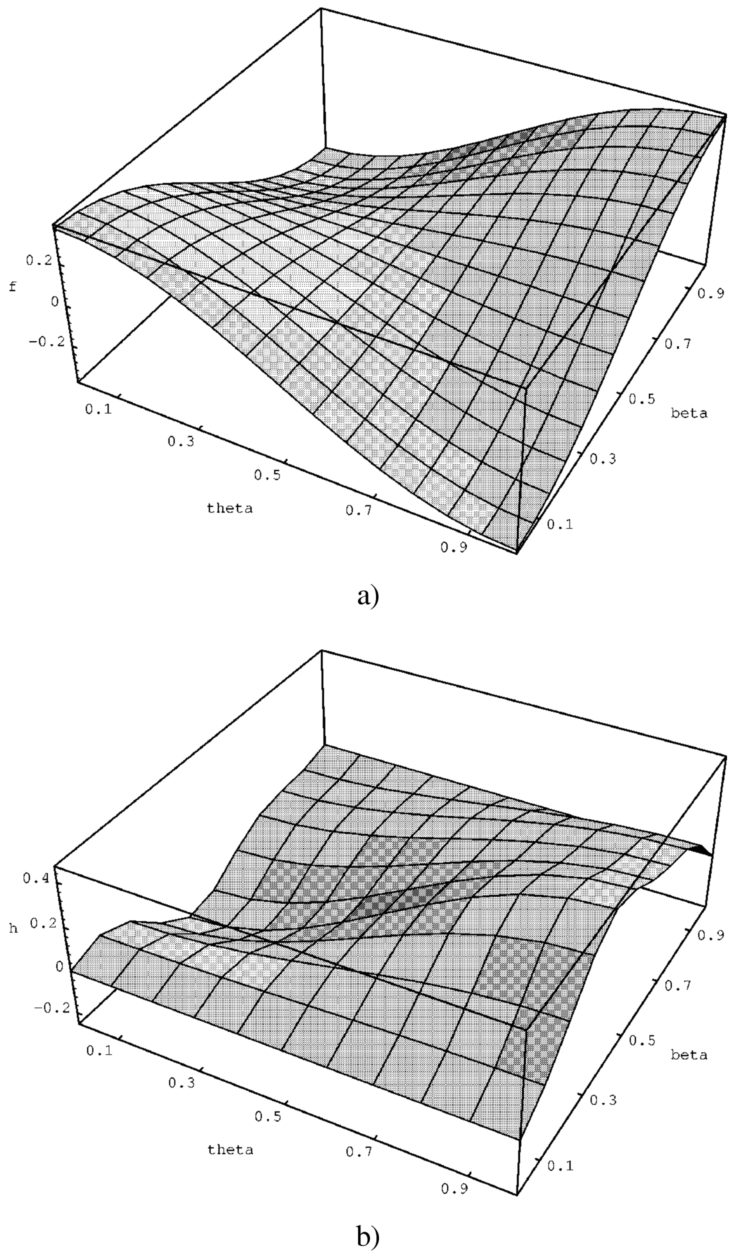


Figure 1. a) The $\theta - \beta$ dependence of the original function $f(g)$ at values of parameters given in the text; b) The $\theta - \beta$ dependence of the solution $h(g)$ for the same parameters. The scale of θ and β axes is given in units of π.

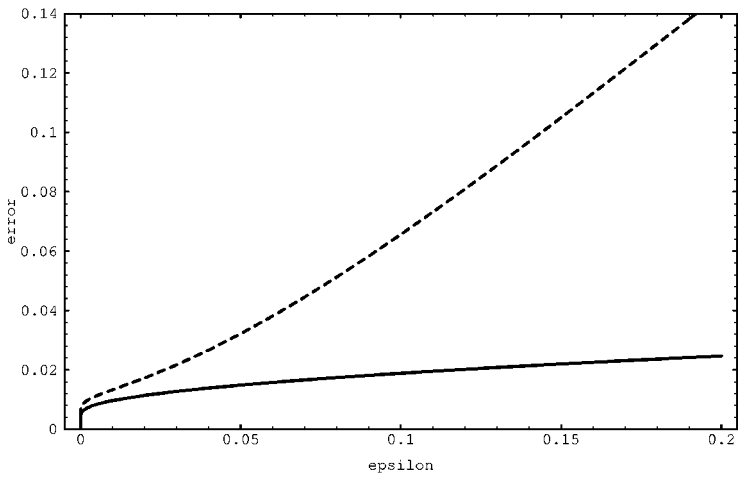


Figure 2. Quadratic error in Example 5.1.1 in Schur's method (solid line) and diagonalization method (dashed line).

5. Regularization Methods and Approximate Solutions

5.1. ON THE EXISTENCE OF SOLUTIONS OF $h * h = f$

For a general square integrable and rapidly decreasing function $f(g)$, even complex solutions of the convolution equation may not exist. This is related to the following reasons.

Apart from the problem of finding the square root of the matrix, which does not always exist (see [15] for discussion of the existence of square roots), we have an additional complication, related to the fact that the matrix elements in the Fourier transform are functions of p. Even for the formally well-defined matrix square root, the inverse Fourier integral may not exist, which indicates that the inverse convolution problem does not have exact solutions. Below we show several typical examples, which illustrate the main reasons why exact solutions do not exists:

- appearance of singularities in the inverse Fourier integral at infinite or finite p (Examples 5.1.1 and 5.1.2 below), or
- absence of exact square roots of the Fourier transform matrix for any or some values of p (Example 5.1.3 below).

Following the examples below, we discuss means of regularization, which allow us to find approximate solutions, when exact solutions do not exist.

EXAMPLE 5.1.1. Let us consider the two-dimensional problem with the function

$$f(\mathbf{r}, A) = \mathrm{e}^{-r^2}(2 + \cos\theta + \cos\phi),$$

where θ is the SO(2) angle, $r = |\mathbf{r}|$ and ϕ is a polar angle of $\mathbf{r}$. The Fourier transform matrix may be computed as ([1])

$$\hat{f} = \begin{bmatrix} \gamma(p) & 0 & 0 \\ \beta(p) & k(p) & \beta(p) \\ 0 & 0 & \gamma(p) \end{bmatrix}, \tag{44}$$

where

$$\gamma(p) = \frac{1}{4}\,\mathrm{e}^{-p^2/4};$$

$$\beta(p) = -\frac{i}{16}\sqrt{\pi}\, p\, {}_1F_1(3/2, 2; -p^2/4);$$

(where ${}_1F_1(a, b; x)$ is a confluent hypergeometric function (see, for example, [16])) and

$$k(p) = \mathrm{e}^{-p^2/4}.$$

The square root of (44) is

$$\hat{f}^{1/2} = \begin{bmatrix} \sqrt{\gamma(p)} & 0 & 0 \\ \frac{\beta(p)}{(\sqrt{\gamma(p)}+\sqrt{k(p)})} & \sqrt{k(p)} & \frac{\beta(p)}{(\sqrt{\gamma(p)}+\sqrt{k(p)})} \\ 0 & 0 & \sqrt{\gamma(p)} \end{bmatrix}. \tag{45}$$

It may be shown that the matrix terms $(\beta(p)/(\sqrt{\gamma(p)}+\sqrt{k(p)}))$ are increasing exponentially with p (it behaves as $(\exp(p^2/8)/p^3)$ for $p \to \infty$) and the inverse Fourier integral does not exist (this problem may not arise with a different choice of r-dependence in the function $f(\mathbf{r}, A)$). One way to regularize this problem would be to replace these matrix elements with the expression

$$\frac{\beta(p)\,\overline{(\sqrt{\gamma(p)} + \sqrt{k(p)})}}{(\epsilon + |\sqrt{\gamma(p)} + \sqrt{k(p)}|^2)},$$

where ϵ is a small positive real parameter. We will discuss how to generalize this approach for arbitrary Fourier transforms in the following subsection.

EXAMPLE 5.1.2. Now we consider three-dimensional function

$$f(g) = f(\mathbf{r}, A) = \mathrm{e}^{-(\mathbf{r}+\mathbf{a})^2} = \mathrm{e}^{-(r^2+a^2+2ra\cos\theta)},$$

where $\mathbf{a} = (0, 0, a)$ and θ is a polar angle of $\mathbf{r}$.

First, we calculate the Fourier transform of this function.

The direct calculation of the Fourier transform would require the computation of an infinite number of Fourier transform matrix elements

$$\hat{f}^{0}_{0,0;l,0}; \quad l = 0, 1, 2, 3, \ldots .$$

We demonstrate, however, that the appropriate left shift reduces the number of harmonics to the single matrix element.

The left shifted function is defined as

$$f^L(g) = f(g'^{-1} \circ g) = \mathrm{e}^{-r^2},$$

where we choose $g' = (\mathbf{a}, I)$, I is an identity matrix (i.e. g' is a pure translation).

The Fourier transform of the left shifted function satisfies the equation

$$\begin{aligned}\hat{f}^L(p) &= \int_{\mathrm{SE}(3)} f(g'^{-1} \circ g)\, \mathcal{U}^\dagger(g, p)\, \mathrm{d}\mu(g)\\ &= \int_{\mathrm{SE}(3)} f(\tilde{g})\, \mathcal{U}^\dagger(g' \circ \tilde{g}, p)\, \mathrm{d}\mu(\tilde{g})\\ &= \left(\int_{\mathrm{SE}(3)} f(\tilde{g})\, \mathcal{U}^\dagger(\tilde{g}, p)\, \mathrm{d}\mu(\tilde{g})\right) \cdot \mathcal{U}^\dagger(g', p)\\ &= \hat{f}(p) \cdot \mathcal{U}^\dagger(g', p),\end{aligned} \tag{46}$$

where $\hat{f}(p)$ is the Fourier transform of $f(g)$. We used in the equation above the change of variables $g'^{-1} \circ g \to \tilde{g}$, the invariance of integration measure, and the group property of matrix elements.

Because this function does not depend on the Euler angles of rotation, only the $\mathcal{U}^0_{0,0;0,0}(\mathbf{r}, A; p)$ element gives a contribution.

Thus, only a single Fourier transform matrix element is nonzero:

$$\begin{aligned}\hat{f}^{L\,0}_{0,0;0,0}(p) &= \int_{\mathrm{SE}(3)} \mathrm{e}^{-r^2}\, \overline{\mathcal{U}^0_{0,0;0,0}(\mathbf{r}, A; p)}\, \mathrm{d}^3 r\, \mathrm{d}A\\ &= 4\pi \int_0^\infty \mathrm{e}^{-r^2}\, \frac{\sin(pr)}{p}\, r\, \mathrm{d}r = (\pi)^{3/2}\, \mathrm{e}^{-p^2/4},\end{aligned}$$

where we have used the fact that $\int_{\mathrm{SO}(3)} \mathrm{d}A = 1$.

Now we may shift the function $f^L(g)$ back, $f(g) = f^L(g' \circ g)$. The analogous Fourier transform equation is

$$\hat{f}(p) = \hat{f}^L(p) \cdot \mathcal{U}(g', p),$$

i.e. it is obtained by inverting (46). We have in matrix form

$$\hat{f}^0_{0,0;l,0}(p) = \hat{f}^{L\,0}_{0,0;0,0}(p) \cdot \mathcal{U}^0_{0,0;l,0}(\mathbf{a}, I; p). \tag{47}$$

We have shown below several matrix elements in explicit form

$$\hat{f}^0_{0,0;0,0} = \pi^{3/2}\, \mathrm{e}^{-p^2/4}\, \frac{\sin pa}{pa};$$

$$\hat{f}^0_{0,0;1,0} = \frac{\sqrt{3}i}{(pa)^2}\, \pi^{3/2}\, \mathrm{e}^{-p^2/4}\, (\sin pa - pa \cos pa);$$

$$\hat{f}^0_{0,0;2,0} = \frac{\sqrt{5}i}{(pa)^3}\pi^{3/2}\,\mathrm{e}^{-p^2/4}\,(3pa\cos pa - 3\sin pa + (pa)^2\sin pa).$$

If we truncate the matrix elements at $l = 1$ then the Fourier transform matrix may be written as

$$\hat{H} = \begin{bmatrix} \hat{f}^0_{0,0;0,0}(p) & 0 & \hat{f}^0_{0,0;1,0}(p) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

(the structure of the matrix is similar if we truncate the Fourier transform for higher l). We denote $\hat{f}^0_{0,0;0,0}(p)$ as $\hat{H}_{11}$ and $\hat{f}^0_{0,0;1,0}(p)$ as $\hat{H}_{13}$.

It is easy to find a formal square root matrix

$$\hat{H}^{1/2} = \begin{bmatrix} \sqrt{\hat{H}_{11}} & 0 & \frac{\hat{H}_{13}}{\sqrt{\hat{H}_{11}}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{49}$$

However, the $\hat{H}_{11}$ matrix element become zero at $p = \pi n/a$, $n = 1, 2, 3, \ldots$. At these points the $\hat{H}_{13}$ element is not equal to zero, which leads to the appearance of a singularities at these points. These singularities are, however, integrable and inverse Fourier integral exists. A similar example:

$$f(h) = f(\mathbf{r}, A) = \mathrm{e}^{-(\mathbf{r}+\mathbf{a})^2} + \mathrm{e}^{-(\mathbf{r}+\mathbf{a_1})^2},$$

where $\mathbf{a} = (0, 0, 1)$ and $\mathbf{a_1} = (0, 0, 1/2)$, has the same structure of the Fourier transform matrix and the square root matrix as in (48) and (49). The singularities appear at $p = 2\pi n$ ($n = 1, 2, 3, \ldots$), because the $\hat{H}_{11}$ matrix element become zero at these points, while the $\hat{H}_{13}$ element is not equal to zero. The singularities become non-integrable for $p = 2\pi n$ ($n = 1, 3, 5, \ldots$) in this case and the inverse Fourier integral does not exists. This means that the exact solution of the convolution equation does not exist in this case.

The other consequence of the singularities may be seen if we try to diagonalize the Fourier transform matrix, take the square root of the diagonalized matrix and transform the matrix back. The matrix (48) may be diagonalized by the transformation matrix

$$T = \begin{bmatrix} 1 & 0 & -\frac{H_{13}}{H_{11}\sqrt{1+|H_{13}/H_{11}|^2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{1+|H_{13}/H_{11}|^2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{50}$$

The inverse matrix is given by

$$T^{-1} = \begin{bmatrix} 1 & 0 & \frac{H_{13}}{H_{11}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{1+ \mid H_{13}/H_{11} \mid^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{51}$$

At $p = 2\pi$, for example, the T^{-1}_{13} and T^{-1}_{33} matrix elements become singular, i.e. the inverse matrix is not defined. If we do not take a square root of the diagonalized matrix, this singularity would be cancelled by the appropriate small (zero) eigenvalues. This cancellation is not possible here after taking the square root of the eigenvalues.

EXAMPLE 5.1.3. In the case when the matrix can be brought by a unitary transformation to the triangular matrix of the form

$$\begin{aligned} U^{\dagger}\, \hat{F}\, U = S \; &= \begin{bmatrix} a(p) & 0 & 0 & 0 \\ 0 & 0 & c(p) & 0 \\ 0 & 0 & 0 & c(p) \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= c(p)\, \{J_0(a(p)/c(p)) \oplus J_3(0)\} \end{aligned} \tag{52}$$

no solutions of the matrix Equation (29) exist. We note that this matrix is already in the Jordan normal form (right side of the equation) if we divide it by $c(p)$ (let us assume that $a(p)/c(p)$ is not singular everywhere). We use the notation $J_n(\lambda)$ for a Jordan block of dimension n with eigenvalues λ on the diagonal. It may be proven that no square roots of matrix (52) exist, because it contains the single Jordan block $J_3(0)$ with zero eigenvalues (see [15]).

Another example with the same problem is a band-limited function with Fourier transform matrix (three dimensional case, only $s = 0$ block)

$$\hat{F}^0 = c(p) \cdot \begin{bmatrix} 0 & i & 0 & -i \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 \end{bmatrix},$$

where $c(p)$ is a real function. We assume that $c(p)$ gives a well defined inverse Fourier integral. Because relations (19) are satisfied, this Fourier transform gives a real function. It is easy to check that this Fourier transform matrix may be brought to the Jordan normal form (we factor out $c(p)$):

$$c(p)\, J_4(0) = T\, \hat{F}^0\, T^{-1}(p) = c(p) \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

by nonsingular transformation $T(p)$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}.$$

$J_4(0)$ is a Jordan block with zero diagonal matrix elements of dimension four, so it may be proven that no square roots of this matrix exist (see [15]).

5.2. REGULARIZATION METHODS FOR FINDING APPROXIMATE SOLUTIONS

We need to develop regularization methods in cases when exact solutions of the convolution equation do not exist. The absence of exact solutions may exhibit itself in the appearance of singularities in the inverse Fourier integral, or the Fourier matrix may not have an exact square root for some p. We propose below two methods of regularization which allow approximate solutions in such cases, i.e approximate solutions $h(g)$ which approximate the original function $f(g)$ with good accuracy in the sense of quadratic norm

$$\int_G |(h * h)(g) - f(g)|^2 \,\mathrm{d}\mu(g) \ll \int_G |f(g)|^2 \,\mathrm{d}\mu(g). \tag{53}$$

A Regularization Method Using the Schur Decomposition

As was clear from Examples 5.1.1 and 5.1.2, singularities appear when the denominator of matrix elements of the Fourier transform becomes zero or approaches zero as $p \to \infty$ faster than the numerator. Both these cases may be regularized by adding the small parameter $\epsilon > 0$ to the denominator in the following way

$$\frac{x}{y} \to \frac{x\,\overline{y}}{(\epsilon + |y|^2)}.$$

We suggest a generalization of this procedure for the matrix case. Any square matrix $\hat{F}$ may be brought by the unitary transformation U to upper triangular form (Schur decomposition) [19]

$$\hat{F}(p) = U(p)\,S(p)\,U^{\dagger}(p),$$

where $S(p)$ is upper triangular. Because the matrix $U(p)$ is unitary, the matrix elements of $U(p)$ are bounded for any p ($|U_{ij}(p)| \leqslant 1$). Moreover, because the function $f(g)$ is square integrable, it may be shown, using the Parseval identity, that each matrix element $S_{ij}(p)$ is square integrable as a function of p. We assume below that the matrix elements of $\hat{F}(p)$ (and, therefore, of $S(p)$) are rapidly decreasing functions of p at $p \to \infty$.

A method of finding the square root R ($R^2 = S$) of the upper triangular matrix has been proposed in [17, 18]. The matrix elements of R (which is also upper triangular) may be found using the algorithm in [18] as follows:

$$R_{ii} = S_{ii}^{1/2} \tag{54}$$

(both branches of the square root are allowed) for diagonal elements, and

$$R_{ij} = \frac{S_{ij} - \sum_{k=i+1}^{j-1} R_{ik}\, R_{kj}}{R_{ii} + R_{jj}}, \quad i < j \tag{55}$$

for superdiagonal elements. The matrix elements may be computed one superdiagonal at a time. This algorithm gives exact solutions (complex, in general) if $R_{ii}(p) + R_{jj}(p) \neq 0$ for any $i,\ j$ at any p.

We propose the algorithm which regularizes (55) at singular point where $R_{ii} + R_{jj} \approx 0$ (at some or any p). We discretize the p interval (for functions rapidly decreasing at infinity we may take a finite interval for p), decompose the matrix $\hat{F}$ and find a square root R of the triangular matrix S using (54) for diagonal elements and the regularized expression (where $\epsilon > 0$)

$$R_{ij} = \frac{(S_{ij} - \sum_{k=i+1}^{j-1} R_{ik}\, R_{kj})\, (\overline{R_{ii} + R_{jj}})}{(\epsilon + |R_{ii} + R_{jj}|^2)}, \quad i < j \tag{56}$$

for the superdiagonal elements. Then the approximate square root solution $\hat{H}$ may be received by the unitary transformation

$$\hat{H} = U R U^{\dagger}. \tag{57}$$

This method is accurate if $R_{ii}(p) + R_{jj}(p)$ are not equal to zero almost everywhere except at a finite (or countably infinite) number of points. It works also in cases when $R_{ii}(p) + R_{jj}(p)$ asymptotically approach zero for $p \to \infty$. The addition of ϵ regularizes the singularities in both cases. For sufficiently small ϵ this procedure does not change significantly the matrix elements for $R_{ii}(p)+R_{jj}(p) \neq 0$. Moreover, the diagonal elements of R are not effected by ϵ. This guarantees that this method provides a good approximation for the matrix $\hat{H}$ in the sense of the Hilbert–Schmidt norm (i.e. the condition (53) is satisfied). We estimate the accuracy of the method for practical example in the next section.

We have to note that the solution provided by this method is not unique. First, each square root may have two branches. We, therefore, may choose different branches of the square root of eigenvalues $R_{ii}(p)$ for different p. If, however, we require additional conditions on smoothness of derivatives of the solution (for example, minimizing the functional which contains the term

$$\int_G \left| r\frac{\partial h}{\partial r} \right|^2 \mathrm{d}\mu(g) = \int_0^{\infty} \left\| p\frac{\mathrm{d}\hat{H}}{\mathrm{d}p} \right\|_2^2 p\, \mathrm{d}p$$

in the two-dimensional case [1], see [2] for three-dimensional operational properties), we have to switch square root to the other branch at branching points of $R_{ii}(p)$ as a function of p, where the eigenvalue $S_{ii}(p)$ switches its value from positive to negative (or from negative to positive) in order to minimize the contribution of this term. Another restriction is imposed by the requirement that the solution must be real. The matrix elements of the Fourier transform satisfy the symmetry relations (18) or (19). To have a real solution we have to take the branches of square root of (m, m) and $(-m, -m)$ $(m \neq 0)$ which conserve the symmetry relations (18) (the same is valid in the three-dimensional case). Particularly, the eigenvalues of solution must appear in complex conjugate pairs to have a solution real (according to Theorem 3.1.1).

Moreover, we have to note that the solution given by (57) is not the most general solution, the matrix $\hat{H} = UXRX^{-1}U^{\dagger}$ is also a solution of the Equation (29) (in analogy with (34)), if X is a matrix which commutes with matrix S (it may not commute with R) [15]. We investigated only the case when X is proportional to a unit matrix.

Because in general the solution provided by this method is not real, we may take a real part of the solution as an approximate real solution, or, what is more preferable, restore the symmetry relations (18) or (19) among the matrix elements of the Fourier transform. This gives a real approximation to the solution and allows us to easily control the deviation of $(h * h)(g)$ from $f(g)$. We may calculate only 'half' of the matrix elements of $\hat{H}$ ($N(N-1)/2$ elements and (0, 0) element in the two-dimensional matrix of dimension $N = 2M + 1$) and receive the other matrix elements according to (18) or (19) (we have to choose elements which provide the smallest quadratic error). If we have some additional symmetries among the matrix elements we may uniquely define other methods to restore symmetry (18) or (19).

This method may not work well for band-limited functions with only few non-zero harmonics (when $R_{ii}(p) + R_{jj}(p)$ equals zero for any p and $R_{km} \neq 0$ for $k = i, j;\ m > k$). We have to use analytical methods (see [15]) for finding the square root in these cases, or add small contribution to the diagonal elements of S and apply the algorithm described above, as in the example below.

EXAMPLE 5.2.1. Let us consider the matrix in Example 5.1.3 when the Fourier matrix may be transformed to the form (52). No square root of the matrix exists. We may apply, however, the algorithm (54), (56), which gives the approximation to the square root matrix as

$$\begin{bmatrix} \sqrt{a(p)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{58}$$

However, it may not be a good approximation if the modulus of $c(p)$ is large. We may improve this approximation if we add the small elements $\epsilon_1(p)$ (for example $\epsilon_1(p) = \epsilon_1 e^{-p^2}$) to the zero diagonal elements of (52) (it does not change a Hilbert–Schmidt norm of the matrix much). After this addition we may apply algorithms (54) and (56). The presence of the parameter ϵ guarantees that matrix elements of the square root do not become large when the small function $\epsilon_1(p)$ appears in the denominator in (56).

A particular choice of the values of the parameter ϵ (and possible additional parameter such as ϵ_1 in Example 5.2.1) and branches of the square root is dictated by the several factors. First, we have to take the values of the parameters and pick up the branches of the square root which minimize the functional

$$C = \int_G |(h * h)(g) - f(g)|^2 \mathrm{d}\mu(g),$$

or the more complicated functional like

$$C = \int_G (|(h * h)(g) - f(g)|^2 + \nu |h(g)|^2 + \rho |\nabla_{\mathbf{r}} h(g)|^2)\, \mathrm{d}\mu(g),$$

where $\nu > 0$ and $\rho > 0$ are penalty parameters, and $\nabla_{\mathbf{r}}$ is a gradient with respect to translation. Also, we may choose the values of the parameters and square root branches, which give the best solution from the point of view of some additional 'physical' or design principles.

Algorithm for Multiple Convolutions

We note that using the approach of [17, 18] the algorithm (54), (56) may be easily generalized for the solution of the equation

$$R^n = S, \tag{59}$$

where S is an upper triangular matrix. The convolution equation for multiple n-fold convolutions may be written as

$$(h * h * \cdots * h)(g) = f(g).$$

After taking the Fourier transform it has the form of matrix equation

$$\hat{H}^n(p) = \hat{F}(p). \tag{60}$$

Using Schur's decomposition

$$\hat{F} = USU^\dagger$$

this equation may be reduced to (59).

The following algorithm gives a regularized solution of (59). We introduce auxiliary matrices

$$R^2 = R_2; \quad R^3 = R_3; \quad \ldots \quad R^{n-1} = R_{n-1}$$

and use notations $R_1 \equiv R$ and $R_0 = \mathbf{1}$, where $\mathbf{1}$ is a unit matrix. Then the solution for diagonal elements is

$$(R_l)_{ii} = (S_{ii})^{l/n}, \quad \text{for } l = 1, \ldots, n-1$$

(all n branches of $S_{ii}^{l/n}$ are allowed, but only one is chosen).

For superdiagonal elements we have equation

$$\begin{aligned}(R_1)_{ij} \;&=\; \left(S_{ij} - \sum_{l=1}^{n-1} (R_{n-l-1})_{ii} \sum_{k=i+1}^{j-1} (R_1)_{ik}\,(R_l)_{kj}\right) \times \\ &\times \frac{\overline{\left(\sum_{l=0}^{n-1} (R_{n-l-1})_{ii}\,(R_l)_{jj}\right)}}{\left(\epsilon + |\sum_{l=0}^{n-1} (R_{n-l-1})_{ii}\,(R_l)_{jj}|^2\right)}, \quad i < j \end{aligned} \tag{61}$$

and equations for auxiliary matrices

$$\begin{aligned}(R_l)_{ij} \;&=\; (R_1)_{ii}\,(R_{l-1})_{ij} + (R_1)_{ij}\,(R_{l-1})_{jj} + \\ &+ \sum_{k=i+1}^{j-1} (R_1)_{ik}\,(R_{l-1})_{kj}, \quad i < j, \end{aligned} \tag{62}$$

for $l = 2, \ldots, n-1$. Equations (61) and (62) allow us to find the matrix elements of R_l (for each l) one superdiagonal at a time. The presence of parameter $\epsilon > 0$ in (61) guarantees the regularization of possible singularities which may come from zeros of $\sum_{l=0}^{n-1} (R_{n-l-1})_{ii}\,(R_l)_{jj}$.

The solution of (60) is given by

$$\hat{H} = U R_1 U^{\dagger}. \tag{63}$$

We note that (63) is not the most general solution. The continuous dependence on arbitrary matrix function $X(p)$ may appear in (63) (in analogy with (34)). We put $X(p) = \mathbf{1}$ in our algorithm.

Regularization Using Diagonalization

We also propose another method of regularization which uses the diagonalization of the matrix.

Because not any matrix may be diagonalized, this method is less general that the previous method. At values of p where the precise Fourier matrix may be brought only to the Jordan normal form [15], the square root calculated according to

$$\hat{H} = T\,\sqrt{A}\,T^{-1} \tag{64}$$

(where T is a transformation matrix and $\sqrt{A}$ is a diagonal matrix with the diagonal values equal to the square root of eigenvalues) gives an error. This, error, however, may be small, if, for example, the measure of points p, where the matrix cannot be diagonalized, is negligible. As we will see in the example of Section 6 the accuracy of this method is comparable with the accuracy of the Schur decomposition method for small ϵ and it is worse for larger ϵ. If the matrix has only few harmonics (with many zero elements for all p), the method may be inaccurate.

As we saw in Example 5.1.2, even for Fourier matrices which may be diagonalized everywhere, the absence of the exact solutions exhibits itself in the appearance of the singularities in matrix elements of T^{-1} at finite p (or as $p \to \infty$). To find a regularized solution we propose to use a pseudoinverse matrix instead of singular inverse matrix

$$\hat{H} = T\sqrt{A}\,T^{\dagger}\,(TT^{\dagger} + \epsilon\mathbf{1})^{-1}, \tag{65}$$

where ϵ is a small positive parameter and $\mathbf{1}$ is a unit matrix. For n-fold convolutions we have to take the $A^{1/n}$ power of the diagonal matrix.

To get a real solution we restore the symmetry in $\hat{H}$ using the symmetry relations (18) (or (19)).

5.3. EXAMPLE OF FINDING THE APPROXIMATE REGULARIZED SOLUTION

We find here the regularized solution of Example 5.1.1 where no exact real or complex solution of the equation exists due to the divergence of the inverse Fourier integral. We used both regularization methods and we restored the symmetry (18) in order to have a real solution. We use ‘half’ of the matrix (to restore the other ‘half’) which gives a smaller quadratic error. In Figure 2 we depicted the quadratic error

$$\frac{\int_{\mathrm{SE}(2)} |(h*h)(g) - f(g)|^2 \mathrm{d}\mu(g)}{\int_{\mathrm{SE}(2)} |f(g)|^2 \mathrm{d}\mu(g)}, \tag{66}$$

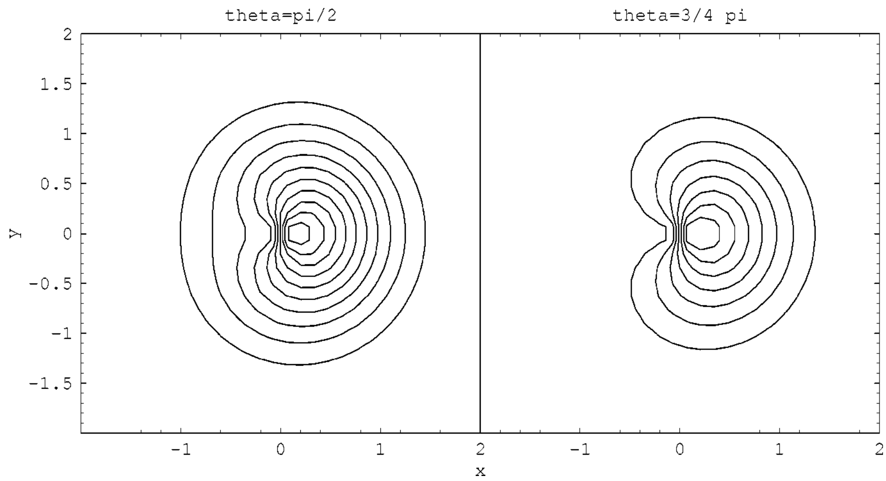
where $h(g)$ is the approximation to the solution of the convolution equation, calculated by the method (54) and (56) (solid line), and by method using diagonalization (dashed line). We see that the first method gives a smaller error (for the same ϵ) than the method which uses diagonalization.

We cannot choose a value of ϵ as small as we wish, because the solution develops a singularity when $\epsilon \to 0$. We may start from some value of ϵ and approach zero value. When the solution starts to exhibit ‘unpleasant’ behavior further reduction of ϵ must be stopped. Another method to fix a value of ϵ is to impose some additional conditions. We may require ϵ to minimize a value of the functional

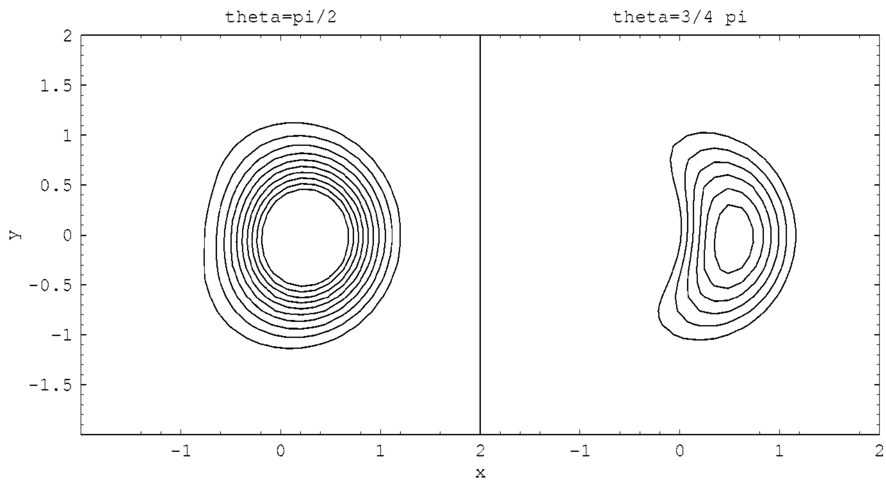
$$C = \int_{\mathrm{SE}(2)} (|(h*h)(g) - f(g)|^2 + \nu|h(g)|^2)\,\mathrm{d}\mu(g) \tag{67}$$

for some value of ‘cost’ parameter ν. For example, for $\nu = 0.1$ the functional (67) is minimized for $\epsilon = 0.07$ in Schur’s decomposition method. For this value of ϵ quadratic error is 1.68% in Schur’s method.

As an example, we depict the contour plot of the solution found in Schur’s method for particular values of SO(2) angle θ in Figure 3b (we take only the first



a)



b)

Figure 3. a) The contour plot of the original function of Example 5.1.1; b) The approximate regularized solution.

(positive) branch of the square root of the eigenvalues, which are positive for all p). We depicted contour lines in the range from 0.1 to 3 spaced with with values 0.32, the original function at the same parameters is depicted on Figure 3a. We note also that the solution has negative regions at $\theta \gtrsim 3/4\pi$ and $\phi \approx \pi$.

We also plot in Figure 4 the convolution $(h * h)(g)$, where $h(g)$ is an approximate solution. We see that this approximation deviates from the original solution at $r \approx 0$, because we change the asymptotic of the solution at $p \to \infty$. For less divergent examples we receive more accurate approximation. The peak of the

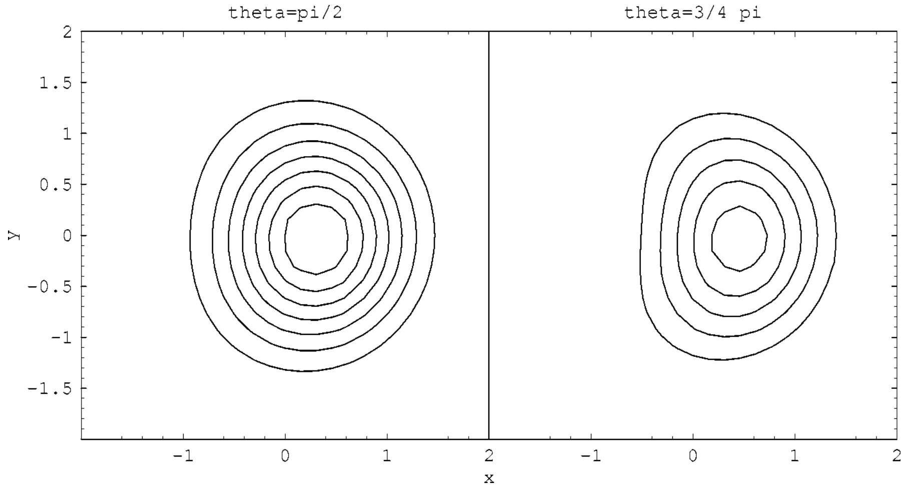


Figure 4. The plot of $(h * h)(g)$, where $h(g)$ is approximate solution of convolution equation.

convolution is ‘flat’ compare with the peak of the original function, the quadratic deviation, however, is small.

Conclusion

In this paper we apply Harmonic Analysis on the motion group to the solution of a nonlinear convolution equation. We investigated cases when no solutions of the convolution equation exist and we suggested numerical regularization methods and algorithms for search of the approximate solution in such cases. An explicit example of the solution technique was given and the quadratic error which estimates the deviation of the convolution of the approximate solutions from the original function was calculated. We note that the numerical example provided was motivated by a robotics problem described in [23].

Appendix. Some Useful Definitions

We define the associated Legendre polynomials as

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x). \tag{A-1}$$

The functions $P^l_{mn}(z)$ are given as in [6]

$$\begin{aligned} P^l_{mn}(z) &= \left[\frac{(l-n)!(l+m)!}{(l-m)!(l+n)!}\right]^{1/2} \frac{(1-z)^{\frac{m-n}{2}}(1+z)^{\frac{m+n}{2}}}{2^m(m-n)!} \times \\ &\quad \times {}_2F_1\left(l+m+1, -l+m; m-n+1; \frac{1-z}{2}\right), \end{aligned}$$

where ${}_2F_1$ is the hypergeometric function. This expression is valid for $m - n \geqslant 0$, the corresponding expressions for other possible values of m, n may be received using the properties of the hypergeometric functions and the symmetry properties of $P^l_{mn}(z)$. We note that $P^l_{mn}(z)$ satisfy

$$P^l_{mn}(z) = (-1)^{m+n} P^l_{nm}(z),$$

$$P^l_{mn}(z) = (-1)^{m-n} P^l_{-m,-n}(z),$$

$$P^l_{mn}(z) = P^l_{-n,-m}(z).$$

Spherical functions are defined as

$$Y^m_l(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P^m_l(\cos\theta)\, \mathrm{e}^{im\phi}. \tag{A-2}$$

References

1. Chirikjian, G.: Fredholm integral equation on the Euclidean motion group, *Inverse Problems* **12** (1996), 579–599.
2. Kyatkin, A. and Chirikjian, G. S.: An operational calculus for the Euclidean motion group, Tech. Report RMS-96-1, Johns Hopkins University, Sept. 1996.
3. Ebert-Uphoff, I. and Chirikjian, G. S.: Discretely actuated manipulator workspace generation by closed-form convolution, in: *Proceeding 1996 ASME Design Engineering Technical Conference and Computers in Engineering Conference*, Irvine, California, August 1996.
4. Chirikjian, G. S.: Kinematic synthesis of mechanisms and robotic manipulators with binary actuators, *J. Mech. Design* **117** (1995), 573–580.
5. Ebert-Uphoff, I. and Chirikjian, G. S.: Efficient workspace generation for binary manipulators with many actuators, *J. Robotic Systems* **12**(6) (1995), 383–400.
6. Vilenkin, N. J. and Klimyk, A. U.: *Representation of Lie Group and Special Functions*, Vols. 1–3, Kluwer Acad. Publ., Dordrecht, 1991.
7. Miller, W.: *Lie Theory and Special Functions*, Academic Press, New York, 1968; Some applications of the representation theory of the Euclidean group in three-space, *Comm. Pure Appl. Math.* **17** (1964), 527–540.
8. Talman, J.: *Special Functions*, W. A. Benjamin, Amsterdam, 1968.
9. Gurarie, D.: *Symmetry and Laplacians. Introduction to Harmonic Analysis, Group Representations and Applications*, Elsevier Science Publisher, The Netherlands, 1992.
10. Coleman, A. J.: *Induced Representations with Applications to S_n and $GL(n)$*, Queen's University, Kingston, Ontario, 1966.
11. Mackey, G. W.: *Induced Representations of Groups and Quantum Mechanics*, W. A. Benjamin, New York, Amsterdam, 1968.
12. Gelfand, I. M., Minlos, R. A. and Shapiro, Z. Ya.: *Representations of the Rotation and Lorentz Groups and Their Applications*, Pergamon Press, New York, 1963.
13. Jones, M. N.: *Spherical Harmonics and Tensors for Classical Field Theory*, Research Studies Press Ltd., England, 1985.
14. Groetsch, C. W.: *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind*, Pitman, Boston, 1984.
15. Gantmacher, F. R.: *The Theory of Matrices*, Chelsea, New York, 1959.

16. Erdelyi, A.: *Higher Transcendental Functions*, Bateman Manuscript Project, Vols. 1–3, McGraw-Hill, New York, 1953–55.
17. Parlett, B. N.: A recurrence among the elements of functions of triangular matrices, *Linear Algebra Appl.* **14** (1976), 117–121.
18. Bjorck, A. and Hammarling, S.: A Schur method for the square root of a matrix, *Linear Algebra Appl.* **52/52** (1983), 127–140.
19. Horn, R. A. and Johnson, C. R.: *Matrix Analysis*, Cambridge University Press, Cambridge, New York, 1988.
20. Vilenkin, N. J.: Bessel functions and representations of the group of Euclidean motions, *Uspekhi Mat. Nauk.* **11** (1956), 69–112 (in Russian).
21. Orihara, A.: Bessel functions and the Euclidean motion group, *Tohoku Math. J.* **13** (1961), 66–71.
22. Sugiura, M.: *Unitary Representations and Harmonic Analysis*, 2nd edn, Elsevier Science Publisher, The Netherlands, 1990.
23. Chirikjian, G. S. and Ebert-Uphoff, I.: Numerical convolution on the Euclidean group with applications to workspace generation, *IEEE Trans. on Robotics and Automation* **14**(1) (1998), 123–136.