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Synthesis of Binary Manipulators Using the Fourier Transform on the Euclidean Group[1]

In this paper we apply the Fourier transform on the Euclidean motion group to solve problems in kinematic design of binary manipulators. In recent papers it has been shown that the workspace of a binary manipulator can be viewed as a function on the motion group, and it can be generated as a generalized convolution product. The new contribution of this paper is the numerical solution of mathematical inverse problems associated with the design of binary manipulators. We suggest an anzatz function which approximates the manipulator's density in analytical form and has few free fitting parameters. Using the anzatz functions and Fourier methods on the motion group, linear and non-linear inverse problems (i.e., problems of finding the manipulator's parameters which produce the total desired workspace density) are solved.

1 Introduction

Robotic manipulators are usually constructed of rigid links and actuators, such as motors or hydraulic cylinders. Recently in the design literature, a number of independent efforts, including the authors' previous work, have considered an alternative paradigm in which finite-state actuators are used. Investigation of bistable compliant mechanisms and MEMS devices may also be considered part of this paradigm shift (Opdahl, Jensen, Howell, 1998), (Matoba, Ishikawa, Kim, Muller, 1994).

For actuators with only a finite number of states, as is the case with stepper motors, bistable snapping actuators, or pneumatic cylinders, the resulting robotic arm has a finite number of configurations and can reach only a finite number of frames (positions and orientations). Each frame may be viewed as an element of the Euclidean motion group $SE(N)$[3] (see (Murray et al., 1994) for references on the motion group).

For discretely actuated manipulators the workspace density, which is defined as the number of reachable frames per unit volume of the motion group $SE(N)$ (Ebert-Uphoff and Chirikjian, 1998), determines how accurately a position and orientation can be reached. This density information is an important factor in the kinematic design and motion planning of discretely actuated manipulator arms (Ebert-Uphoff and Chirikjian, 1996).

In previous work, we have shown that the density function for a manipulator with K^n states can be calculated using n generalized convolutions. This is reduced to $\log_2 n$ convolutions for a manipulator composed of a cascade of identical links or platforms. In analogy with Fourier methods for functions on the real line, on the sphere, or on finite groups, which give an efficient way to perform convolutions and allow one to apply Fast Fourier Transform (FFT) methods (Elliot, Rao, 1982), (Driscoll, Healy, 1994), (Rockmore, 1994), the application of Fourier methods on the motion group provides a considerable saving in the computation time when performing the convolutions on the motion group needed to generate manipulator workspaces (Kyatkin and Chirikjian, 1999). In the present work, Fourier methods are used to numerically solve linear and nonlinear inverse problems, i.e., the problems of determining the manipulator's parameters which produce the total desired workspace density.

The mathematical framework for non-commutative Fourier methods on the two and three dimensional motion group with applications to the solution of convolution equations are explained in (Kyatkin and Chirikjian, 1998). In the present work, we implement these methods numerically for the two dimensional motion group and apply these methods to solve inverse problems that arise in binary manipulator design.

In Section 2 we give a general review of the Fourier transform on the 2D motion group. In Section 3 we give an overview of mathematical inverse problems arising in binary manipulator design. In Section 4 we suggest an anzatz function which describes the manipulator's density in an analytical form and has relatively few free parameters. Using anzatz functions and Fourier methods we solve linear and non-linear inverse problems in Sections 5 and 6, respectively.

2 The Fourier Transform on the Motion Group

Here we give briefly the general expressions which define the Fourier transform on the two dimensional motion group. For more complete references for the two dimensional case and for the three dimensional case see (Kyatkin and Chirikjian, 1998).

Each element of $SE(2)$ is parametrized in polar coordinates as:

$$g(r, \phi, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & r\cos\phi \\ \sin\theta & \cos\theta & r\sin\phi \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $r = |\mathbf{r}|$ is the magnitude of the translational part of the motion. The group law is simply matrix multiplication.

The inner product of square-integrable functions on the motion group is given by

$$(f, h) = \int_{SE(2)} \overline{f(g)} h(g) d\mu(g). \quad (1)$$

The norm of a complex-valued function on the group is $|f(g)| = \sqrt{(f, f)}$. The quadratic error of one function relative to another is defined as

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[3] $SE(N)$ denotes "special Euclidean group," which is the group of rigid motions of N-dimensional Euclidean space. We refer to this group simply as the "motion group."

$$q = \frac{\int_{SE(2)} |f(g) - f'(g)|^2 d\mu(g)}{\int_{SE(2)} |f(g)|^2 d\mu(g)} \quad (2)$$

The invariant integration measure on $SE(2)$ is given by

$$d\mu(g(r, \phi, \theta)) = \frac{1}{(2\pi)^2} r dr d\phi d\theta$$

We need to use unitary representations of the motion group, $SE(2)$,[4] to generate the Fourier transform on the two dimensional motion group. If we use these representations then the Fourier transform of the convolution of functions may be written as the matrix product of the Fourier transform of each function (see below), which gives considerable savings in computation time.

A number of works including (Vilenkin, 1956), (Orihara, 1961), (Talman, 1968) have shown that the unitary irreducible representation matrices for $SE(2)$, which are denoted as $\mathscr{U}(g, p)$, have entries given by

$$u_{mn}(g(r, \phi, \theta), p) = i^{n-m} e^{-i[n\theta + (m-n)\phi]} J_{n-m}(pr) \quad (3)$$

for $m, n \in \mathbf{Z}$, where $J_\nu(x)$ is the ν^{th} order Bessel function, $g \in SE(2)$, and p is a continuous parameter which enumerates the representations, in analogy with the Fourier transform parameter on the real line.

Definition. For any integrable complex-valued function $f(g)$ on the motion group $G = SE(2)$ we define the Fourier transform as

$$\mathscr{F}(f) = \hat{f}(p) = \int_G f(g) \mathscr{U}(g^{-1}, p) d\mu(g)$$

where $g \in G$.

The inverse Fourier transform is used to reconstruct a function from its Fourier transform as:

$$f(g) = \mathscr{F}^{-1}(\hat{f}) = \int_0^\infty \mathrm{Tr}\,(\hat{f}(p)\mathscr{U}(g, p)) p dp \quad (4)$$

Plancherel equality. The Plancherel (or generalized Parseval) equality for square-integrable functions on the motion group $G = SE(2)$ is:

$$\int_G |f(g)|^2 d\mu(g) = \int_0^\infty \|\hat{f}(p)\|_2^2 p dp$$

where $\|\hat{f}(p)\|_2^2$ is the square of the Hilbert-Schmidt norm

$$\|\hat{f}\|_2 = \mathrm{Tr}\,\sqrt{(\hat{f}\hat{f}^\dagger)},$$

$\hat{f}^\dagger$ is a Hermitian conjugate of $\hat{f}$, and Tr is the trace.

Convolution property. The Fourier transform of the convolution of two square-integrable functions is the product of the Fourier transforms of the functions. The convolution of functions on the motion group is defined as

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1} \circ g) d\mu(h), \quad (5)$$

and the application of the Fourier transform yields

$$\mathscr{F}(f_1 * f_2) = \mathscr{F}(f_2)\mathscr{F}(f_1), \quad (6)$$

where now the product is a matrix product of the Fourier transform matrices

$$\mathscr{F}(f_1 * f_2)_{mn} = \sum_{k=-\infty}^{\infty} \hat{f}^2_{mk} \hat{f}^1_{kn}$$

where $\hat{f}^i_{mn} = \mathscr{F}(f_i)_{mn}$. We note that now the order of the product of Fourier transforms matters. In practice, this product is truncated at $k = \pm M$, where M is a chosen finite number.

3 Inverse Problems in Binary Manipulator Design

The following inverse problems arise naturally in the area of kinematic design of binary manipulators:

- Given a final desired workspace density and known parameters for the lower half of a manipulator find the kinematic parameters of the upper half of the manipulator such that the total workspace density of the manipulator fits the desired density in the best way (in the sense of the quadratic deviation). This is the so-called linear inverse problem.
- Given a final desired workspace density, find the kinematic parameters of each half of the manipulator which result in a workspace density which fits the desired density in the best way. This is a non-linear inverse problem.

The problems above may be written respectively as linear and non-linear integral convolution equations on the motion group. The linear inverse problem may be formulated as

$$(\alpha * \beta)(g) = \int_{SE(2)} \alpha(h)\beta(h^{-1} \circ g) d\mu(h) = \gamma(g), \quad (7)$$

where $\gamma(g)$ is the desired total workspace density, $\alpha(g)$ is the density of the lower part of manipulator and $\beta(g)$ is the unknown density of the upper part.

The non-linear inverse problem may be written as

$$(\alpha * \alpha)(g) = \int_{SE(2)} \alpha(h)\alpha(h^{-1} \circ g) d\mu(h) = \gamma(g), \quad (8)$$

where $\gamma(g)$ is the total desired density, and $\alpha(g)$ is the unknown density of both the lower and upper part (assuming the manipulator is homogeneous).

The direct way of solving these problems, i.e., fitting the total density to a given desired density for each value of manipulator kinematic parameters, would require multiple convolutions (one for each set of parameters) and it would be very costly computationally (since one could imagine doing hundreds of iterations before the minimum is found). The problem becomes easier if we want to fit the density of only a few modules to the given function. So we have to reduce the problem to the fitting problem for a small number of modules which may be computed by brute force. Because the total density, which describes the workspace density of whole manipulator, may be written as the convolution of the densities of manipulator segments, the problem becomes simpler in Fourier space, where the convolution is just a matrix product of Fourier transform matrices.

The first kind of problem, (7), leads to the linear matrix equation of the type

$$\hat{\beta}(p)\hat{\alpha}(p) = \hat{\gamma}(p) \quad (9)$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ denote the Fourier transform matrices of the lower, upper, and whole desired workspace density.

For a nonsingular matrix $\hat{\alpha}$ the solution is straight forward:

$$\beta(g) = \mathscr{F}^{-1}(\hat{\gamma}(p)\hat{\alpha}^{-1}(p)), \quad (10)$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform.

When the matrix $\hat{\alpha}$ is singular the problem may be reduced to the problem of minimization of an appropriate quadratic functional for chosen values of regularization parameters. For the simple functional

[4] See (Vilenkin, Klimyk, 1991), (Talman, 1968), (Vilenkin, 1956), (Orihara, 1961), (Sugiura, 1990), and (Chirikjian and Kyatkin, 2000) for definitions

$$C = \int_{SE(2)} (|(\alpha*\beta)(g) - \gamma(g)|^2 + \epsilon|\beta(g)|^2 + \nu(\beta(g), -\nabla_r^2\beta(g)))d\mu(g)$$

the solution may be expressed in Fourier space as

$$\hat{\beta} = \hat{\gamma}\hat{\alpha}^{\dagger}(\hat{\alpha}\hat{\alpha}^{\dagger} + (\epsilon + \nu p^2)\mathbb{1})^{-1}, \tag{11}$$

where 1 is an identity matrix, ϵ and ν are regularization parameters, ∇_r^2 denotes the Laplacian with respect to translation, and $\dagger$ denotes Hermitian conjugate. In fact, we have to choose the parameters ϵ and ν as small as possible in order to minimize the quadratic error. But the parameters cannot be taken arbitrarily small because the solution starts to exhibit oscillatory and singular behavior, and the quadratic norm of the solution increases as parameters approach to zero. In fact, there is an approximate range of "boundary" values where the norm of the solution is of the order of norm of the $\alpha(g)$, for these values further reduction of the parameters must be stopped. The quadratic error, however, does not depend strongly on the exact values of parameters, so we may pick any small values of parameters in this region.

4 Analytical Description of Workspace Density with an Anzatz Function.

Before we start to discuss the inverse non-linear problem we have to find an appropriate way to describe the desired workspace densities in analytical form. From previous work (Chirikjian and Ebert-Uphoff, 1998) we observe that the density is "shrinking" with increasing orientation angle θ. At the same time it is "rotating" in the $x - y$ plane with increasing θ and "moves" closer to the origin. We also may characterize the density by the point of maximal value of density for each fixed orientation angle. We want to incorporate these important "global" features of the workspace density into an anzatz[5] function which has these properties and has a relatively small number of free parameters. For "symmetric" manipulators (that is, ones with no preferred bending direction) the workspace density also has a symmetry

$$\theta \to -\theta, \quad \phi \to -\phi$$

To describe symmetric workspaces for manipulators with a relatively small number of modules ($n_{\text{mod}} \lesssim 10$) we suggest to parametrize the density functional as

$$f'(r, \phi, \theta) = c_5\left(\frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{(r - x)^2}{2\sigma_1^2}\right)\right. \times (1 + c_3 \cos\theta)^n \left.\frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(-\frac{(\phi - c_4\theta)^2}{2\sigma_2^2}\right)\right) \tag{12}$$

where $x = \frac{1}{2}(1 + \cos\theta)c_1 + \frac{1}{2}(1 - \cos\theta)c_2$. The Gaussian term (normal distribution) centered at x describes the radial dependence of the workspace density. We choose $c_2 < c_1$, so the center of the distribution moves closer to the origin with increasing θ. The term containing c_3 describes "shrinking" of the workspace with increasing θ. The power n is some positive number. We assume that for $c_3 \le 1$ an allowable range of θ values is $|\theta| \le \arccos(-1/c_3)$, and $f'(g) = 0$ for θ outside of this range. The term containing the ϕ-dependence is responsible for the "rotation" of the workspace with increasing θ. We illustrate the anzatz function parameters in Fig. 1.

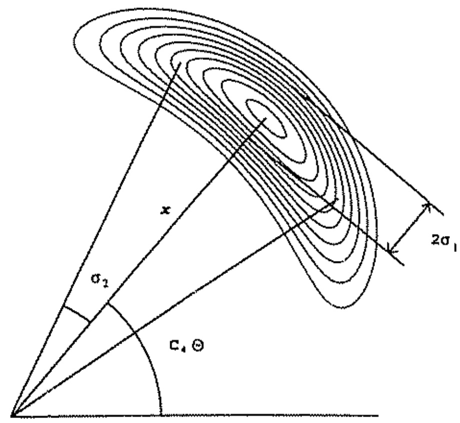

Fig. 1 Parameters describing the model

We assume that $-\pi < \theta \le \pi$, $-\pi < \phi \le \pi$, and the density function is assumed to be 2π-periodic.

We have to mention that considerable deviations from the anzatz may appear for $\theta \approx \pi$ (this is where "disconnected" regions in each θ-slice of the density function may appear from counting the configurations which differ by multiples of 2π in orientation angle θ, i.e., for angles which are outside the range $-\pi < \theta \le \pi$). For a manipulator with a large number of modules ($n \gtrsim 10$) we suggest replacing the last term

$$\frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(-\frac{(\phi - c_4\theta)^2}{2\sigma_2^2}\right)$$

with the term

$$\left[\frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(-\frac{(\phi - c_4\theta)^2}{2\sigma_2^2}\right) + h(\theta)\left(\frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(-\frac{(\phi + c_4'\theta)^2}{2\sigma_2^2}\right)\right)\right] \tag{13}$$

where $c_4' = \frac{1}{2}(1 - \cos\theta)c_4 + \frac{1}{2}(1 + \cos\theta)c_6$, and $h(\theta)$ is an even function such that $h(\pi) = h(-\pi) = 1$.

We note that the c_5 coefficient must be determined from the condition

$$\int_{SE(2)} f(g)d\mu(g) = K^n$$

where K^n is the total number of configurations of the manipulator. For convenience we divide the density function by K^n, so the function is normalized on 1. The anzatz function describes the high-density region of workspace where approximately 90% of states are located.

As an example, consider a six-module variable-geometry-truss manipulator with leg length and base width parameters $l_{\min} = 0.12$, $l_{\max} = 0.2$, $s = 0.2$. The workspace may be approximated by the anzatz function (12) with the parameters

$$c_1 = 0.71; \quad \sigma_1 = 0.09; \quad c_2 = 0.53; \quad c_3 = 1.0;$$
$$c_4 = 0.47; \quad \sigma_2 = 0.38; \quad c_5 = 8.98; \quad n = 1.1.$$

The error of the approximation in the sense of the quadratic norm (2) is $q = 24.0\%$.

We may also describe the desired workspace using the anzatz function (12) or (13) and choosing the appropriate coefficients for this function.

[5] Anzatz is a term often used in the physical sciences when empirical observations are used to form a model in the absence of a well established physical principle.

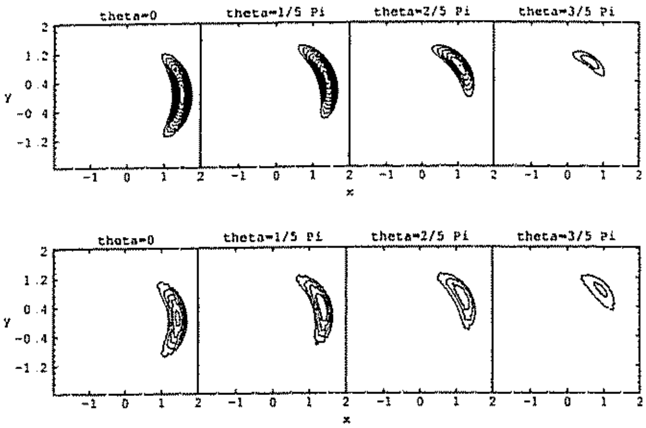

Fig. 2 Desired and calculated density functions for the whole manipulator

5 Example Solution of the Linear Inverse Problem.

As an example, we solved the linear inverse problem for the following anzatz function (with the modified term (13)) with the coefficients

$$c_1 = 1.5; \quad c_2 = 0.9; \quad c_3 = 0.5;$$
$$c_4 = 0.5; \quad c_6 = 2.5; \quad \sigma_1 = 0.15;$$
$$\sigma_2 = 0.46 \quad n = 2.0; \quad c_5 = 4.09;$$

and the choice of $h(\theta) = \exp(-(|\theta| - 2.83)^2/0.1)$, for $|\theta| \leq 2.83$; and $h(\theta) = 1$, for $|\theta| > 2.83$. We note that the particular choice of $h(\theta)$ does not change considerably the norm of the function, because it affects only the regions of low density The anzatz function with the above parameters describes the desired density function $\gamma(g)$. We choose the manipulator density $\alpha_{mnp}(g)$ to be described by the workspace of the six module manipulator with the parameters

$$l_{\min} = 0.12; \quad l_{\max} = 0.20; \quad s = 0.20.$$

The anzatz function for the given values of parameters is depicted in Fig. 2(a).

We look for solutions using (11), where we put $\nu = 0$. The solution exhibits singular behavior if $\epsilon \to 0$. We choose the value of $\epsilon = 0.01$ (the norm of the solution is $|\beta|^2 = 30.79$). The corresponding approximate solution $\beta(g)$, found according to (11), is depicted in Fig. 3(a). We note that the solution is not strictly positive. We truncated the negative part of the solution in the fitting procedure (the norm of positive part of the solution is $|\beta_{pos}|^2 = 26.34$).

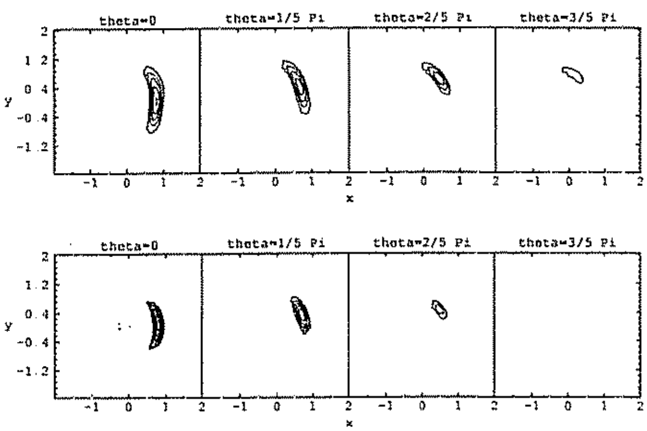

Fig. 3 Calculated density functions for half of the manipulator

Fitting of the six module manipulator density $\beta_{mnp}(g)$ to the function $\beta(g)$ by brute force in the space of three parameters $l_{\min}$, $l_{\max}$, s (the fitting was performed at approximately 120 "points" of the parameter's space in 2 hours) gives the following values of the parameters which produce the first five smallest quadratic intermediate errors q_{int} (deviation of $\beta_{mnp}(g)$ from $\beta(g)$):

$l_{\min}$	$l_{\max}$	s	q_{int}	q
0.12	0.25	0.19	55.8%	29.4%
0.12	0.24	0.19	55.9%	26.1%
0.12	0.24	0.20	56.2%	26.8%
0.12	0.24	0.21	56.0%	27.6%

The error increases rapidly for other parameters, for $l_{\min} = 0.15$; $l_{\max} = 0.23$; $s = 0.19$ it is 173%. We compared the Fourier approximation of the desired anzatz function $\gamma(g)$ with the final density $(\alpha_{mnp} * \beta_{mnp})(g)$ for these values of the parameters and give values of the quadratic error q in the table. We show the convolution $(\alpha_{mnp} * \beta_{mnp})(g)$ for the smallest quadratic error q and

$$l_{\min} = 0.12; \quad l_{\max} = 0.24; \quad s = 0.19 \tag{14}$$

in Fig. 2(b) (computed by the Fourier method with $M = 4$). We note that the calculated solution is an approximate one. The exact solution (value of the parameters of manipulator) is located in the vicinity of these values. If better accuracy is desired, direct fitting of $(\alpha_{mnp} * \beta_{mnp})(g)$, found by the Fourier method, may be performed for the parameters in the vicinity of (14). We found that $(\alpha_{mnp} * \beta_{mnp})(g)$ for the parameter values of $\beta_{mnp}(g)$

$$l_{\min} = 0.13; \quad l_{\max} = 0.22; \quad s = 0.19;$$

approximates $\gamma(g)$ with the best accuracy (in this case the error was 14.0%). The direct fitting of the convolution is, however, at least 7 times more costly computationally (for each "point" in the space of manipulator parameters) for $M = 4$, and more than 30 times for $M = 15$. Thus, the solution of the linear inverse problem (11) gives a fast way to find an approximate density function $\beta(g)$ with acceptable error.

6 Example Solution of the Nonlinear Inverse Problem

The solution of the nonlinear problem may be found in a similar fashion, i.e., we find first numerically the approximate solution of the nonlinear convolution equation and then find the manipulator's parameters which describe the workspace density with the smallest quadratic error.

The non-linear problem becomes a problem of a search for the "square root" of a matrix in Fourier space, i.e., the solution must satisfy the equation

$$\sum_k \hat{\alpha}(p)_{mk}\hat{\alpha}(p)_{kn} = \hat{\gamma}(p)_{mn}$$

for each value of p.

An algorithm for the approximate solution of the non-linear problem, which uses Schur decomposition of Fourier matrices, was described in (Kyatkin and Chirikjian, 1998). Again, the solutions depend on the regularization parameter ϵ. We have to choose the value of the parameter ϵ, which gives the value of the norm in the region $|\alpha|^2 \lesssim 50$ (for a six module manipulator). Moreover, we have additional continuous arbitrariness of the solution related to the following fact The square root of a Fourier matrix requires one to take the square roots of eigenvalues of the Fourier matrix $\hat{\gamma}(p)$. The square root of each eigenvalue has two branches (positive and negative branches for real positive numbers), and we may take different branches for different values of the parameter p (see (Kyatkin and Chirikjian, 1998) for details) While it does not change the norm of the solution, it changes considerably the shape of the function. From the direct convolution of anzatz functions we may observe that the convolution of anzatz functions produces an anzatz-like function for some values of parameters. Thus, we require the "square root" solution to be anzatz-like (i.e., it may be approximated by an anzatz function with a small error). This condition may be implemented in Fourier space as follows. First, we choose the branches of the square root of the eigenvalues using the "trial" anzatz function, i.e., we convolve the trial function with itself and choose a branch by comparing the square root of eigenvalues of the convolution with eigenvalues of the trial function. Then, we use this prescription for the branches of the square root to find the solution of the non-linear problem for the given desired total workspace density.

As an example we solve the non-linear problem for the following set of coefficients of the anzatz function (with term (13)) which describes the desired total workspace density

$$c_1 = 1.6; \quad c_2 = 0.95; \quad c_3 = 0.5;$$

$$c_4 = 0.5; \quad c_6 = 2.5; \quad \sigma_1 = 0.15;$$

$$\sigma_2 = 0.45 \quad n = 2.0; \quad c_5 = 3.85; \qquad (15)$$

We use a value of the regularization parameter $\epsilon = 0.03$. We prescribe a branch of the square root of the eigenvalues based on trial function which has the coefficient values $\frac{1}{2}c_1$, $\frac{1}{2}c_2$, and $\frac{1}{2}\sigma_1$ The other coefficient values for the trial function are on the same order as those in (15).

This should yield a function similar to the solution, because the value of c_1 is scaled approximately by a factor of two when two anzatz functions are convolved. Using this prescription we found numerically the solution of the non-linear problem which is depicted in Fig. 3(b). Because the prescription is only approximate, small deviations from anzatz-like shape appears (such as regions with negative values). These, however, do not affect considerably the norm of the solution.

Fitting of the six module manipulator density $\alpha_{mnp}(g)$ to the function $\alpha(g)$ performed by brute force in the space of three parameters $l_{\min}$, $l_{\max}$, s gives the following values of the parameters which minimizes the quadratic error (first three local minima with the smallest values of the errors are shown)

$$l_{\min} = 0.12; \quad l_{\max} = 0.24; \quad s = 0.21; (58.1\%) \qquad (16)$$

$$l_{\min} = 0.13; \quad l_{\max} = 0.24; \quad s = 0.23; (59.3\%) \qquad (17)$$

$$l_{\min} = 0.13; \quad l_{\max} = 0.23; \quad s = 0.21; (59.6\%) \qquad (18)$$

The error increases rapidly for other values, for example for $l_{\min} = 0.15$; $l_{\max} = 0.22$; $s = 0.20$ it is 236.7%. We computed the value of the quadratic deviation of the convolved function $(\alpha_{mnp} * \alpha_{mnp})(g)$ from the desired function determined by the coefficients (15), and found that the minimum (18) has a smallest quadratic error 18.6% (the minimum (17) gives 26.8% of error, and (16) gives 47.0%). The contour plot of the desired function and the manipulator density for the parameters in (18) are shown in Fig. 4(a)–(b) for $\theta = 0$.

If better accuracy is desired the direct fitting of $(\alpha_{mnp} * \alpha_{mnp})(g)$ (computed by the Fourier convolution method) to the desired function $\gamma(g)$ may be performed for the parameter values in the vicinity of (18). We found that for the manipulators parameters

$$l_{\min} = 0.14; \quad l_{\max} = 0.23; \quad s = 0.24;$$

the manipulator workspace density has a smallest deviation from $\gamma(g)$ equals 14.6%. All convolutions were performed using $M = 4$.

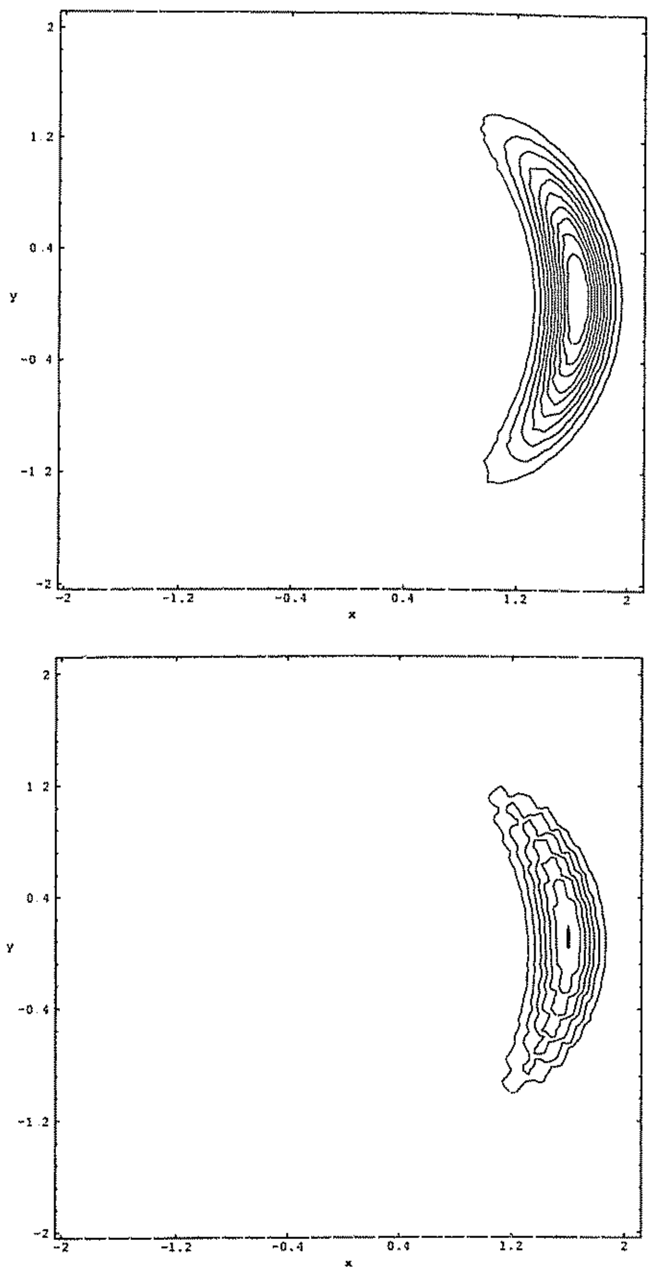

Fig. 4 Comparison of desired density function and solution of the non-linear problem

Conclusions

In this paper we suggested an anzatz, which allows one to describe the workspace densities of the two dimensional manipulators in a simple analytic form with few free parameters. Anzatz functions and Fourier methods on the motion group allow one to solve the linear and non-linear inverse problems of kinematic design with error in the range 20–30%. Workspace density synthesis, Fourier methods, and convolution methods were implemented in the C programming language. Part of the matrix computations in the inverse problems were performed using Mathematica 2.2 programs.

We note that Fourier methods for the three dimensional motion group and analytical examples of Fourier transforms and solutions of linear and non-linear inverse problems are described in (Kyatkin and Chirikjian, 1998; Chirikjian and Kyatkin, 2000), though the application of these methods to 3-D manipulator design is an open and challenging numerical problem.

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