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WORKSPACE GENERATION AS A DIFFUSION PROCESS

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ABSTRACT

In this paper we show that the workspace of a highly articulated manipulator can be found by solving a partial differential equation. This diffusion-type equation describes the evolution of the workspace density function depending on manipulator length and kinematic properties. The support of the workspace density function is the workspace of the manipulator. The PDE governing workspace density evolution is solvable in closed form using the Fourier transform on the group of rigid-body motions. We present numerical results that use this technique.

INTRODUCTION

Consider a highly articulated robot arm with macroscopically serial structure. That is, the arm may either be serial or consist of serially stacked platforms. For a continuously-actuated manipulator with n degrees of freedom, each sampled at K values, K^n positions and orientations in the workspace result. Such discretizations of continuous-motion manipulators has been considered in the literature. See, for example, (6; 9). For reviews of other techniques in the analysis of manipulators and workspaces see (1; 2; 10).

This discrete collection of reachable positions and orientations can be described using a probability density function $\rho(g)$ where $g \in SE(3)$ is a frame of reference. If $\rho_i(g)$ is the density function of the i^{th} segment (joint or platform) in the manipulator with n segments, then (3):

$$\rho(g) = (\rho_1 * \rho_2 * \cdots * \rho_n)(g)$$

where $*$ denotes convolution of functions of motion, which is defined as (4):

$$(\rho_1 * \rho_2)(g) = \int_{SE(3)} \rho_1(h)\rho_2(h^{-1} \circ g)d(h).$$

Here $d(h)$ for $h \in SE(3)$ is the bi-invariant integration measure for $SE(3)$ (see (8)).

In this paper we view the workspace of a highly articulated manipulator as something that grows (or evolves) from a single point source at the base. As we allow the length of the manipulator to increase from zero, the workspace grows into the full volume corresponding to the whole arm. In this way of viewing manipulator workspaces, the density function $\rho(g; L)$ corresponds to a segment of length L. $\rho(g; 0) = \delta(g)$ and $\rho(g; 1)$ is the density of the whole workspace (with the manipulator length normalized to unity). If the manipulator is homogeneous along its length one would expect

$$\rho(g; L_1) * \rho(g; L_2) = \rho(g; L_1 + L_2).$$

In Section 3 we present a diffusion equation with two parameters for planar manipulators: the degree of articulability of the manipulator and the degree of asymmetry. We show how this equation can be solved to find the workspace density function $\rho(g; 1)$. This requires the techniques reviewed in Section 2. Numerical results are presented in Section 4.

FOURIER ANALYSIS OF MOTION

The Euclidean motion group, $SE(N)$ [1], is the semidirect product of $I\!R^N$ with the special orthogonal group, $SO(N)$. We denote elements of $SE(N)$ as $g = (\mathbf{a}, A) \in SE(N)$ where $A \in SO(N)$ and $\mathbf{a} \in I\!R^N$. For any $g = (\mathbf{a}, A)$ and $h = (\mathbf{r}, R) \in SE(N)$, the group law is written as $g \circ h = (\mathbf{a} + A\mathbf{r}, AR)$, and $g^{-1} = (-A^T\mathbf{a}, A^T)$. It is often convenient to think of an element of $SE(N)$ as an $(N+1) \times (N+1)$ homogeneous transformation matrix of the form:

$$g = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

Each element of $SE(2)$ is parameterized in polar coordinates as:

$$g(a, \phi, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & a\cos\phi \\ \sin\theta & \cos\theta & a\sin\phi \\ 0 & 0 & 1 \end{pmatrix}$$

where a is the magnitude of translation.

The Fourier transform of a function of motion, $f(g)$, is an infinite-dimensional matrix defined as (4):

$$\mathcal{F}(f) = \hat{f}(p) = \int_G f(g) U(g^{-1}, p)\, d(g)$$

where $U(g,p)$ is an infinite dimensional matrix function of g and a frequency parameter p with the property that $U(g_1 \circ g_2, p) = U(g_1, p) U(g_2, p)$. This kind of matrix is called a *matrix representation* of $SE(2)$. It has the property that it converts convolutions on $SE(2)$ into matrix products:

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_2)\mathcal{F}(f_1),$$

and the original function can be reconstructed as

$$\mathcal{F}^{-1}(\hat{f}) = f(g) = \int_0^\infty \mathrm{trace}(\hat{f}(p) U(g,p)) p dp.$$

Explicitly, the matrix elements of $U(g,p)$ are expressed as (4):

$$u_{mn}(g(a,\phi,\theta), p) = i^{n-m} e^{-i[n\theta + (m-n)\phi]} J_{n-m}(p\, a) \quad (1)$$

where $J_\nu(x)$ is the ν^{th} order Bessel function.

From this expression, and the fact that $U(g,p)$ is a unitary representation, we have that:

$$u_{mn}(g^{-1}(a,\phi,\theta), p) = u^{-1}_{mn}(g(a,\phi,\theta), p) =$$

$$\overline{u_{nm}(g(a,\phi,\theta), p)} = i^{n-m} e^{i[m\theta + (n-m)\phi]} J_{m-n}(pa). \quad (2)$$

Using the basis for the Lie algebra $se(2)$:

$$X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

one finds

$$g_1(t) = \exp(tX_1) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$g_2(t) = \exp(tX_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix};$$

$$g_3(t) = \exp(tX_3) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Many rigid-body motion in the plane can be expressed as an appropriate combination of these three basic motions.

The way to take partial derivatives of a function of motion (such as a workspace density function) is to evaluate

$$\tilde{X}_i^R f = \frac{d}{dt} f(g \circ \exp(tX_i))|_{t=0}.$$

Explicitly, we can define differential operators $\tilde{X}_i^R$ (in polar coordinates) as:

$$\tilde{X}_1^R = \cos(\theta - \phi)\frac{\partial}{\partial a} + \frac{\sin(\theta - \phi)}{a}\frac{\partial}{\partial \phi}$$

[1] The notation $SE(N)$ comes from the terminology Special Euclidean group of N dimensional space.

$$\tilde{X}_2^R = -\sin(\theta-\phi)\frac{\partial}{\partial a} + \frac{\cos(\theta-\phi)}{a}\frac{\partial}{\partial \phi}$$

$$\tilde{X}_3^R = \frac{\partial}{\partial \theta}.$$

DIFFUSION ON THE MOTION GROUP WITH CLOSED-FORM SOLUTION

Consider the diffusion-type equation

$$\frac{\partial f}{\partial L} = \left(\tilde{X}_2^R + \beta(\tilde{X}_3^R)^2 + \alpha\tilde{X}_3^R\right) f. \quad (3)$$

Equation (3) describes a process that evolves on the group of rigid-body motions. The parameter β describes how flexible the manipulator is in the sense of how much a segment of the manipulator can bend per unit length. If the manipulator can bend a lot, then β will have a large value. If the range of motion is very small, the value of β will be small. The parameter α describes the asymmetry in how the manipulator bends. when $\alpha = 0$, the manipulator can reach left and right with equal ease. When $\alpha < 0$, there is a preference for bending to the right, and when $\alpha > 0$ there is a preference for bending to the left.

This simple two-parameter model qualitatively captures the behavior that has been observed in numerical simulations of workspace densities of discretely-actuated variable-geometry truss manipulators (7).

In analogy with the classical Fourier transform, which converts derivatives of functions of position into algebraic operations in Fourier space, there are operational properties for the motion-group Fourier transform.

By the definition of the $SE(2)$-Fourier Transform $\mathcal{F}$ and operator $\tilde{X}_i^R$, we can have

$$\mathcal{F}[\tilde{X}_i^R f] = u(X_i,p)\hat{f}(p)$$

where

$$u(X_i,p) = \frac{d}{dt}\left(U(\exp(tX_i),p)\right)\Big|_{t=0}.$$

Explicitly,

$$u_{mn}(\exp(tX_1),p) = i^{n-m}J_{m-n}(pt).$$

We know that

$$\frac{d}{dx}J_m(x) = \frac{1}{2}[J_{m-1}(x) - J_{m+1}(x)]$$

and

$$J_{m-n}(0) = \begin{cases} 1 \text{ for } m-n=0 \\ 0 \text{ for } m-n \neq 0. \end{cases}$$

Hence,

$$\frac{d}{dt}u_{mn}(\exp(tX_1),p)\Big|_{t=0} = -\frac{ip}{2}(\delta_{m,n+1} + \delta_{m,n-1}).$$

Likewise,

$$u_{mn}(\exp(tX_2),p) = i^{n-m}e^{-i(n-m)\pi/2}J_{m-n}(pt) = J_{m-n}(pt),$$

and so

$$\begin{aligned}\frac{d}{dt}u_{mn}(\exp(tX_2),p)\Big|_{t=0} &= \frac{p}{2}(J_{m-n-1}(0) - J_{m-n+1}(0)) \\ &= \frac{p}{2}(\delta_{m,n+1} + \delta_{m,n-1}).\end{aligned}$$

Similarly, we find

$$u_{mn}(\exp(tX_3),p) = e^{-imt}\delta_{m,n}$$

and

$$\frac{d}{dt}u_{mn}(\exp(tX_3),p)\Big|_{t=0} = -im\delta_{m,n}.$$

Hence, given an equation of the form of (3), we can convert this to an infinite system of linear ordinary differential equations:

$$\frac{d\hat{f}}{dL} = B\hat{f}$$

where the elements of the matrix

$$B = u(X_2,p) + \beta[u(X_3,p)]^2 + \alpha u(X_3,p)$$

are written explicitly as

$$B_{mn} = \frac{p}{2}(\delta_{m,n+1} - \delta_{m,n-1}) - (\beta m^2 + i\alpha m)\delta_{m,n}.$$

In principle, since $f(g;0) = \delta(g)$, and $\hat{f}(p;0)$ is the identity, we have for $L = 1$ the solution

$$\hat{f}(p;1) = \exp(B),$$

which is substituted in the Fourier inversion formula to recover $f(g;1)$ (which we denote simply as $f(g)$).

In practice, the numerical solution requires the truncation of this infinite system so that we consider a band-limited approximation. The result is then substituted into the Fourier inversion formula for the motion group.

NUMERICAL RESULTS

In numerical implementations, the infinite-dimensional matrix function $U(g,p)$ is truncated. The result is a band-limited approximation. We chose the upper bound of the frequency parameter p to be 250. The matrix $U(g,p)$ is truncated at $-L_B \leq m, n \leq L_B$ where $L_B = 12$. Since the numerical results of the Fourier transform of this diffusion equation are approximated by a band-limited version, the outer elements (values of $\hat{f} = \exp(B)$ with $|m|, |n| \to L_B$) can have larger errors. We therefore impose a second cutoff frequency of $L_B = 3$ after the exponentiation when substituting into the Fourier inverse formula.

The effects of the two parameters α and β on the workspace are shown in Figure 1 and Figure 2 respectively. In Figure 1, α is fixed to 0 and β varies from 2 to 4. The function $f(g)$ at different values of rotation angle θ $(0, \pi/4, \pi/2, 3\pi/4$, and $\pi)$ are given. We see that for larger β, i.e, a more flexible manipulator, the workspace (support of the density function $f(g)$) is larger. In Figure 2, we set β =3, α=1, 3, and -3. Slices of $f(g)$ for several values of the rotation angle θ are given. The positional workspace densities (integral of $f(g)$ over all values of θ) for these six kinds of conditions are illustrated in Figure 3. Figures 3.e and 3.f show the fact the α affects how the workspace of the manipulator bends to the left and right.

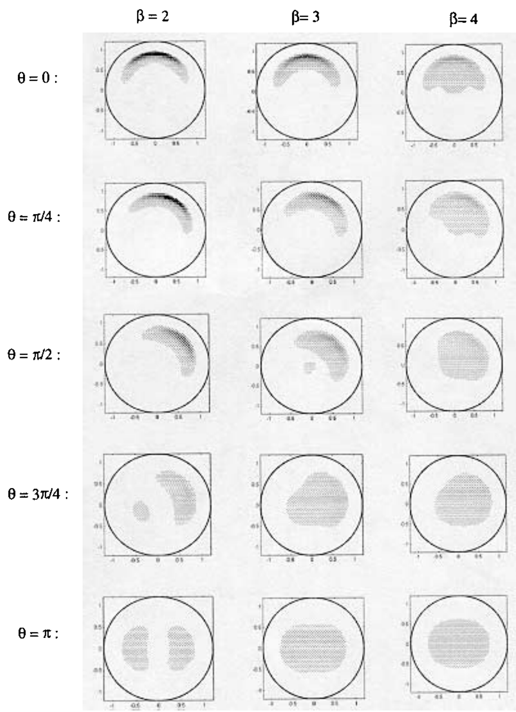

Figure 1 ($\alpha = 0$)

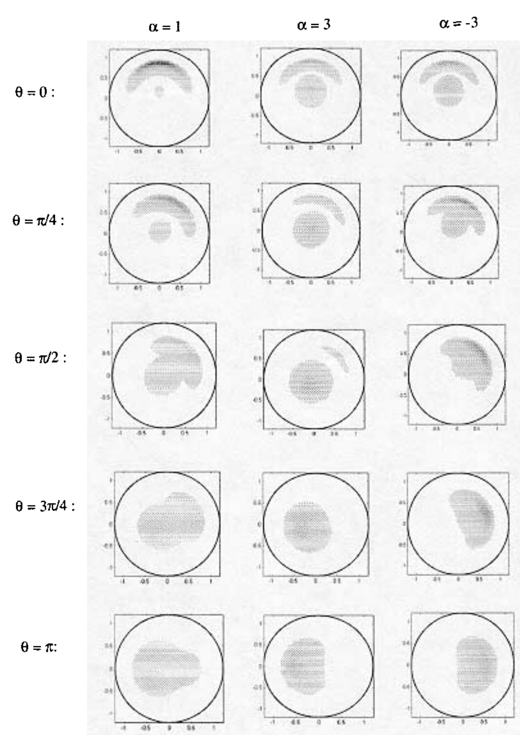

Figure 2 ($\beta = 0$)

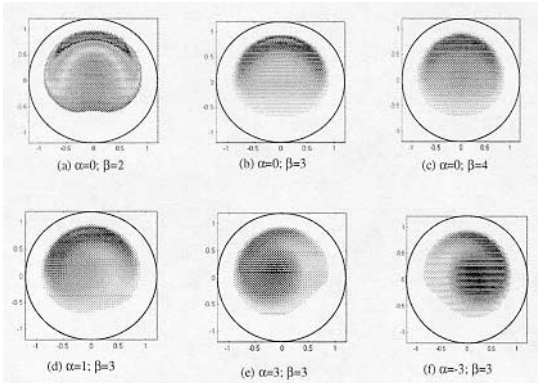

Figure 3

CONCLUSION

In this work, it is shown how the workspace density of frames reachable by a highly articulated manipulator can be generated as the solution to a partial differential equation. This diffusion-type equation describes a process that evolves on the group of rigid-body motions. The support of this function is the workspace. We presented numerical solutions for different kinematic parameters.

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