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A NEW O(N) METHOD FOR INVERTING THE MASS MATRIX FOR SERIAL CHAINS COMPOSED OF RIGID BODIES

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ABSTRACT

Over the past several decades a number of O(n) methods for forward and inverse dynamics computations have been developed in the multibody dynamics and robotics literature. In this paper, a method developed in 1973 by Fixman for O(n) computation of the mass-matrix determinant for a polymer chain consisting of point masses is adapted and modified. In other recent papers, we and our collaborators recently extended this method in order for Fixman's results to be applicable to robotic manipulator models with lumped masses. In the present paper we extend these ideas further to the case of serial chains composed of rigid-bodies. This requires the use of relatively deep mathematics associated with the rotation group, $SO(3)$, and the special Euclidean group, $SE(3)$, and how to differentiate functions of group-valued argument.

Introduction

The first O(n) algorithm for dynamics calculation was developed in the multi-body systems literature by Vereshchagin in 1975 [10]. In the robotics literature, the Luh-Walker-Paul recursive Newton-Euler approach [12] has been a cornerstone of manipulator inverse dynamics for many years. Hollerbach showed that O(n) inverse dynamics could also be achieved within a Lagrangian dynamics setting [11]. In [3] and [4], Rodriguez and coworkers described O(n) solutions for both the forward and inverse dynamics problems for serial manipulators by using recursive techniques from linear filtering and smoothing theory. They also showed two recursive factorization methods of the mass matrix for fixed-base and mobile-base manipulators [4]: Newton-Euler factorization; and Innovations factorization. As another approach, a new decomposition method using analytical Gaussian Elimination (GE) of the inertia matrix was introduced by Saha in [9]. Extending his previous work, in [8], he presented a recursive forward dynamics algorithm for open-loop, serial-chain robots. This work builds on the work of Angeles and Ma who developed the Natural Orthogonal Complement for the manipulator mass matrix [13]. Saha's algorithm has O(n) computational complexity and is also based on reverse GE applied to the analytical expressions of the elements of the inertia matrix. Interestingly, all of these approaches appear to be unaware of developments in the polymer physics literature in which Fixman developed an O(n) method for computing the determinant of a serial chain structure composed of rigid links and point masses [1]. In a series of recent papers, we extended Fixman's method to yield a new method for O(n) inversion of the mass matrix for serial chains consisting of point masses [2]. In the present paper we extend this formulation further by considering chains of rigid bodies.

Fixman's Theorem and Efficient Inversion of the Mass Matrix

We begin by introducing *Fixman's Theorem* to the ASME community and showing how extensions of Fixman's results can be used to efficiently invert the expression $M\mathbf{x} = \mathbf{b}$ where M is the mass matrix for a serial chain composed of point masses. In Section , we extend this to systems composed of rigid bodies.

Fixman's Method

Given a set of N point masses $\{m_1,...,m_N\}$ with corresponding set of absolute positions $\{\mathbf{x}_1,...,\mathbf{x}_N\}$, we define the $3N$-dimensional composite position vector as

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_N \end{pmatrix}$$

Let us assume that n generalized coordinates $q_1,...,q_n$ are used to parameterize $\mathbf{x}$. Then

$$\frac{\partial \mathbf{x}}{\partial \mathbf{q}} = \begin{pmatrix} \frac{\partial \mathbf{x}_1}{\partial q_1} & \cdots & \frac{\partial \mathbf{x}_1}{\partial q_k} & \cdots & \frac{\partial \mathbf{x}_1}{\partial q_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{x}_j}{\partial q_1} & \cdots & \frac{\partial \mathbf{x}_j}{\partial q_k} & \cdots & \frac{\partial \mathbf{x}_j}{\partial q_n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial \mathbf{x}_N}{\partial q_1} & \cdots & \frac{\partial \mathbf{x}_N}{\partial q_k} & \cdots & \frac{\partial \mathbf{x}_N}{\partial q_n} \end{pmatrix} \tag{1}$$

If we define

$$[\mathrm{diag}(m_i)] = \begin{pmatrix} m_0 \mathbb{I} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & m_i \mathbb{I} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & m_n \mathbb{I} \end{pmatrix},$$

where $\mathbb{I}$ now stands for the 3×3 identity matrix, then the mass matrix can be computed as:

$$\mathbf{G} = \left[\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right]^T [\mathrm{diag}(m_i)] \left[\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right]. \tag{2}$$

The elements of $\mathbf{G}$ above can be written in the more familiar form

$$g_{ij} = \sum_{k=1}^{N} m_k \frac{\partial \mathbf{x}_k}{\partial q_i} \cdot \frac{\partial \mathbf{x}_k}{\partial q_j}$$

In the case when no constraints are imposed, $n = 3N$, and all of the matrices in (2) are square. This means that we can use the fomula $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ to get

$$\mathbf{G}^{-1} = \left[\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right]^{-1} [\mathrm{diag}(1/m_i)] \left[\left(\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right)^T\right]^{-1}.$$

But since in general for an invertible matrix $\mathbf{A}$,

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

and since for square Jacobians

$$\left[\frac{\partial \mathbf{x}}{\partial \mathbf{q}}\right]^{-1} = \left[\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right]$$

it follows that

$$\mathbf{G}^{-1} = \left[\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right] [\mathrm{diag}(1/m_i)] \left[\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right]^T. \tag{3}$$

Recall that the derivative of a scalar-valued function of vector-valued argument, $f(\mathbf{z})$ with $\mathbf{z} \in \mathbb{R}^N$, with respect to its argument is a row vector,

$$\frac{\partial f}{\partial \mathbf{z}} = \left[\frac{\partial f}{\partial z_1}, ..., \frac{\partial f}{\partial z_N}\right].$$

This means that

$$\frac{\partial \mathbf{q}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial q_1}{\partial \mathbf{x}_1} & \cdots & \frac{\partial q_1}{\partial \mathbf{x}_k} & \cdots & \frac{\partial q_1}{\partial \mathbf{x}_N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial q_j}{\partial \mathbf{x}_1} & \cdots & \frac{\partial q_j}{\partial \mathbf{x}_k} & \cdots & \frac{\partial q_j}{\partial \mathbf{x}_N} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial q_n}{\partial \mathbf{x}_1} & \cdots & \frac{\partial q_n}{\partial \mathbf{x}_k} & \cdots & \frac{\partial q_n}{\partial \mathbf{x}_N} \end{pmatrix}$$

where each entry in the above matrix is a 3-dimensional row vector. Therefore,

$$\left(\frac{\partial \mathbf{q}}{\partial \mathbf{x}}\right)^T = \begin{pmatrix} \left(\frac{\partial q_1}{\partial \mathbf{x}_1}\right)^T & \cdots & \left(\frac{\partial q_j}{\partial \mathbf{x}_1}\right)^T & \cdots & \left(\frac{\partial q_n}{\partial \mathbf{x}_1}\right)^T \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \left(\frac{\partial q_1}{\partial \mathbf{x}_k}\right)^T & \cdots & \left(\frac{\partial q_j}{\partial \mathbf{x}_k}\right)^T & \cdots & \left(\frac{\partial q_n}{\partial \mathbf{x}_k}\right)^T \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \left(\frac{\partial q_1}{\partial \mathbf{x}_N}\right)^T & \cdots & \left(\frac{\partial q_j}{\partial \mathbf{x}_N}\right)^T & \cdots & \left(\frac{\partial q_n}{\partial \mathbf{x}_N}\right)^T \end{pmatrix}$$

Hence, multiplying out the matrices in (3), one finds the elements of $\mathbf{G}^{-1}$ are of the form

$$g^{ij} = (\mathbf{G}^{-1})_{ij} = \sum_{k=1}^{N} \frac{1}{m_k} \left(\frac{\partial q_i}{\partial \mathbf{x}_k}\right) \left(\frac{\partial q_j}{\partial \mathbf{x}_k}\right)^T. \tag{4}$$

With this background, we can state the following:

Fixman's theorem

Given a chain of $N+1$ point masses with internal coordinates $(c_1, ..., c_{3(N+1)})$ partitioned into f soft variables and r hard variables as $(a_1, ..., a_f; b_1, ..., b_r)$ such that $f + r = 3(N+1)$, then

$$g_{ij} = \sum_{l=0}^{N} m_l (\partial \mathbf{x}_l / \partial c_i) \cdot (\partial \mathbf{x}_l / \partial c_j)$$

and

$$g^{ij} = (G^{-1})_{ij} = \sum_{l=0}^{N} \frac{1}{m_l} (\partial c_i / \partial \mathbf{x}_l) \cdot (\partial c_j / \partial \mathbf{x}_l)$$

where $\mathbf{x}_l$ is the absolute position of the l^{th} point mass (and the dot product of row vectors above is defined in the obvious way).

The above matrices can be partitioned as

$$G = \begin{pmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{pmatrix}$$

and

$$G^{-1} = \begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix}.$$

Using the fact that $GG^{-1} = I$, and hence

$$G \begin{pmatrix} I & H_{ab} \\ 0 & H_{bb} \end{pmatrix} = \begin{pmatrix} G_{aa} & 0 \\ G_{ba} & I \end{pmatrix}, \tag{5}$$

it follows that

$$(\det G)(\det H_{bb}) = \det G_{aa}.$$

As we shall see, for serial chains this provides a very useful tool to compute $\det G_{aa}$ (where $G_{aa} = M$ is the mass matrix for a constrained system, such as a robotic manipulator arm with mass lumped at the joints and constraints on link lengths). The usefulness comes from the fact that $(\det G)$ can be computed in closed form and that $(\det H_{bb})$ can be computed efficiently for serial structures because the serial structure makes H_{bb} a bandlimited matrix. Examples of the application of this method to computing the determinant of mass matrices in $O(n)$ computations can be found in the polymer literature, where the method was introduced [1].

Extending Fixman's Method to Compute M^{-1}

Let us now denote the set of soft variables with the subscript '1' and hard variables with subscrip '2' (rather than Fixman's notation of 'a' and 'b'). We now consider the fast inversion of the equation

$$G_{11}\mathbf{x} = \mathbf{b} \tag{6}$$

where G_{11} is the mass matrix for a serial chain with constraints (i.e., the mass matrix M known in the robotics literature). In contrast, G is the full mass matrix when motion in all coordinates is allowed. The direct numerical inversion of (6) uses $O(n^3)$ computations since G_{11} in general is a full matrix.

Our approach will be to solve the larger system of equations:

$$\begin{pmatrix} G_{11} & 0 \\ G_{12}^T & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}. \tag{7}$$

Obviously, if we can solve this system, then we can solve the original.

Recall that by definition

$$G^{-1} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}.$$

Hence, by multiplying block-by-block, we see that

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} G_{11} & 0 \\ G_{12}^T & \mathbb{I} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & H_{12} \\ 0 & H_{22} \end{pmatrix}.$$

This is essentially the same as Fixman's linear algebra trick. Viewed in a slightly different way, this can be written as:

$$\begin{pmatrix} G_{11} & 0 \\ G_{12}^T & \mathbb{I} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{I} & H_{12} \\ 0 & H_{22} \end{pmatrix}^{-1} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}.$$

which means we can solve (7) if we can efficiently compute the above matrices. In fact, we can write

$$\begin{pmatrix} \mathbb{I} & H_{12} \\ 0 & H_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbb{I} & -H_{12}H_{22}^{-1} \\ 0 & H_{22}^{-1} \end{pmatrix}.$$

This means that (7) can be inverted as

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & -H_{12}H_{22}^{-1} \\ 0 & H_{22}^{-1} \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}. \tag{8}$$

Performing all of the block multiplications, and extacting $\mathbf{x}$ (since we do not care about $\mathbf{y}$), we find

$$\mathbf{x} = (H_{11} - H_{12}H_{22}^{-1}H_{12}^T)\mathbf{b}. \tag{9}$$

Now if we were to explicitly compute all of the matrices above and mutiply and add them together, this would not be particularly fast. However, we can compute $\mathbf{c} = H_{11}\mathbf{b}$ and $\mathbf{d} = H_{12}^T\mathbf{b}$ efficiently since most of the entries in H_{ij} are zeros for chains with many degrees of freedom. $\mathbf{e} = H_{22}^{-1}\mathbf{d}$ can be computed efficiently by decomposing H_{22} into LU form. Since H_{22} is bandlimited, so too will be L and U, and the evaluation of $\mathbf{e}$ can be performed in $O(n)$ computations. Finally, $\mathbf{f} = H_{12}\mathbf{e}$ can again be computed in $O(n)$ as can $\mathbf{x} = \mathbf{c} - \mathbf{f}$.

Extension to Rigid Bodies

While Fixman's theorem represents a clever insight into how to directly exploit the serial nature of a chain consisting of point masses, the mathematics involved is nothing more than multivariable calculus. This is because the positions of point masses are quantities that belong to $\mathbb{R}^3$, and taking gradients in this space is a common mathematical operation. In contrast, it is not at all clear without invoking higher mathematics how to do the same for rigid bodies. In other words, whereas it makes sense to compute gradients of the form $\partial/\partial\mathbf{x}_i$ where $\mathbf{x}_i \in \mathbb{R}^3$, and the unconstrained Jacobian $\partial\mathbf{x}/\partial\mathbf{q}$ in Equation 1 is square, when considering rigid bodies, would it mean anything to compute $\partial/\partial R_i$ where $R_i \in SO(3)$? And the dimensions of the associated Jacobians would certainly not be square given that rotation matrices have nine elements and only three free parameters. Hence, in this section we address how to compute derivatives in an appropriate way for functions of rotations and rigid-body motions in order to extend Fixman's approach.

To begin, recall that if R is a rotation matrix,

$$\frac{d}{dt}\left(R^T R\right) = \frac{d}{dt}(\mathbb{I}) = 0,$$

and so

$$R^T\dot{R} = -\dot{R}^T R = -(R^T\dot{R})^T.$$

Due to the skew-symmetry of this matrix, we can write $\omega = \text{vect}(R^T\dot{R})$. The vector ω is the angular velocity as seen in the body-fixed frame of reference.

The kinetic energy of a rigid-body is then

$$T = \frac{1}{2}m\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + \frac{1}{2}\omega^T I\omega$$

where I is the constant moment of inertia matrix as seen in the body-fixed frame with origin at the center of mass, and $\mathbf{x}$ is the position of the center of mass of the rigid body as seen in a space-fixed frame of reference.

The following subsections develop the mathematical framework needed to handle the rotational contribution to kinetic energy in our extension of Fixman's theorem.

Jacobians Associated with Parameterized Rotations

When a time-varying rotation matrix is parameterized as

$$R(t) = A(q_1(t), q_2(t), q_3(t)) = A(\mathbf{q}(t)),$$

then by the chain rule from calculus, one has

$$\dot{R} = \frac{\partial A}{\partial q_1}\dot{q}_1 + \frac{\partial A}{\partial q_2}\dot{q}_2 + \frac{\partial A}{\partial q_3}\dot{q}_3.$$

Multiplying on the left by R^T and extracting the dual vector from both sides, one finds that

$$\omega = J(A(\mathbf{q}))\dot{\mathbf{q}} \tag{10}$$

where

$$J(A(\mathbf{q})) = \left[\text{vect}\left(A^T\frac{\partial A}{\partial q_1}\right), \text{vect}\left(A^T\frac{\partial A}{\partial q_2}\right), \text{vect}\left(A^T\frac{\partial A}{\partial q_3}\right)\right].$$

For ZXZ Euler angles, α, β, and γ, this is written explicitly as [18]

$$J = [R_3(-\gamma)R_1(-\beta)\mathbf{e}_3, R_3(-\gamma)\mathbf{e}_1, \mathbf{e}_3] = \begin{pmatrix} \sin\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta\cos\gamma & -\sin\gamma & 0 \\ \cos\beta & 0 & 1 \end{pmatrix}. \tag{11}$$

Differential Operators for $SO(3)$

Let $A \in SO(3)$ be an arbitrary rotation, and $f(A)$ be a function which assigns a real or complex number to each value of A. In analogy with the definition of the partial derivative (or directional derivative) of a complex-valued function of $\mathbb{R}^N$-valued argument, we can define differential operators which act on functions of rotation-valued argument:

$$\frac{\partial}{\partial \xi_{\mathbf{n}}} f(A) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[f(A \cdot \mathrm{ROT}[\mathbf{n},\varepsilon]) - f(A)\right] = \frac{d f(A \cdot \mathrm{ROT}[\mathbf{n},t])}{dt}\Big|_{t=0}. \tag{12}$$

In the above definition the variable ξ is introduced to emphasize that the derivative is not with respect to $\mathbf{n}$, but rather the derivative along a coordinate defined by the direction $\mathbf{n}$.

Note that for small motions,

$$\mathrm{ROT}[\mathbf{n},\theta] \approx \mathbb{I} + \theta N = \mathbb{I} + \theta(n_1E_1 + n_2E_2 + n_3E_3)$$

where

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}; \quad E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and $\mathrm{vect}(E_i) = \mathbf{e}_i$.

We now find the explicit forms of the operators $\frac{\partial}{\partial \xi_{\mathbf{n}}}$ in any 3-parameter description of rotation $A = A(q_1,q_2,q_3)$. Expanding in a Taylor series, one writes

$$\frac{\partial f}{\partial \xi_{\mathbf{n}}} = \sum_{i=1}^{3} \frac{\partial f}{\partial q_i}\frac{\partial q_i^r}{\partial \varepsilon}\Big|_{\varepsilon=0}$$

where $\{q_i^r\}$ are the parameters such that $A(q_1,q_2,q_3)\mathrm{ROT}[\mathbf{n},\varepsilon] = A(q_1^r,q_2^r,q_3^r)$. The 'r' denotes the fact that each q_i is perturbed by multiplication on the right by $\mathrm{ROT}[\mathbf{n},\varepsilon]$.

The coefficients $\frac{\partial q_i^r}{\partial \varepsilon}|_{\varepsilon=0}$ are determined by observing two different-looking, though equivalent, ways of writing $A \cdot \mathrm{ROT}[\mathbf{n},\varepsilon]$ for small ε:

$$A + \varepsilon AN \approx A \cdot \mathrm{ROT}[\mathbf{n},\varepsilon] \approx A + \varepsilon \sum_{i=1}^{3} \frac{\partial A}{\partial q_i}\frac{\partial q_i^r}{\partial \varepsilon}\Big|_{\varepsilon=0}.$$

We then have that

$$N = \sum_{i=1}^{3} A^T \frac{\partial A}{\partial q_i}\frac{\partial q_i^r}{\partial \varepsilon}\Big|_{\varepsilon=0},$$

or

$$\mathbf{n} = \mathrm{vect}(N) = \sum_{i=1}^{3} \mathrm{vect}\left(A^T \frac{\partial A}{\partial q_i}\right)\frac{\partial q_i^r}{\partial \varepsilon}\Big|_{\varepsilon=0},$$

which is written as $\mathbf{n} = J\frac{d\mathbf{q}^r}{d\varepsilon}|_{\varepsilon=0}$. This allows us to solve for

$$\frac{d\mathbf{q}^r}{d\varepsilon}\Big|_{\varepsilon=0} = J^{-1}\mathbf{n}.$$

Recall that J is the "body" Jacobian calculated in (11) for the ZXZ Euler angles. Its inverse is

$$J^{-1} = \begin{pmatrix} \sin\gamma/\sin\beta & \cos\gamma/\sin\beta & 0 \\ \cos\gamma & -\sin\gamma & 0 \\ -\cot\beta\sin\gamma & -\cot\beta\cos\gamma & 1 \end{pmatrix}.$$

Making the shorthand notation $\frac{\partial}{\partial \xi_{\mathbf{e}_i}} = \frac{\partial}{\partial \xi_i}$, we then write for the ZXZ Euler angles

$$\frac{\partial}{\partial \xi_1} = -\cot\beta\sin\gamma\frac{\partial}{\partial \gamma} + \frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial \alpha} + \cos\gamma\frac{\partial}{\partial \beta};$$

$$\frac{\partial}{\partial \xi_2} = -\cot\beta\cos\gamma\frac{\partial}{\partial \gamma} + \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial \alpha} - \sin\gamma\frac{\partial}{\partial \beta};$$

$$\frac{\partial}{\partial \xi_3} = \frac{\partial}{\partial \gamma}.$$

Infinitesimal Motions and Associated Jacobians

For "small" motions the matrix exponential description of a rigid-body motion is approximated well when truncated at the first two terms:

$$\exp\left[\begin{pmatrix} \Omega & \mathbf{v} \\ \mathbf{0}^T & 0 \end{pmatrix}\Delta t\right] \approx \mathbb{I} + \begin{pmatrix} \Omega & \mathbf{v} \\ \mathbf{0}^T & 0 \end{pmatrix}\Delta t. \tag{13}$$

Here $\Omega = -\Omega^T$ and $\mathrm{vect}(\Omega) = \omega$describe the rotational part of the displacement. Since the second term in (13) consists mostly of zeros, it is common to extract the information necessary to describe the motion as

$$\begin{pmatrix} \Omega & \mathbf{v} \\ \mathbf{0}^T & 0 \end{pmatrix}^{\vee} = \begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix}.$$

This six-dimensional vector is called an *infinitesimal* screw motion or *infinitesimal twist*.

Given a homogeneous transform

$$H(\mathbf{q}) = \begin{pmatrix} R(\mathbf{q}) & \mathbf{b}(\mathbf{q}) \\ \mathbf{0}^T & 1 \end{pmatrix}$$

parameterized with $(q_1,...,q_6)$, which we write as a vector $\mathbf{q} \in \mathbb{R}^6$, one can express the homogeneous transform corresponding to a slightly changed set of parameters as the truncated Taylor series

$$H(\mathbf{q}+\delta\mathbf{q}) = H(\mathbf{q}) + \sum_{i=1}^{6} \Delta q_i \frac{\partial H}{\partial q_i}(\mathbf{q}).$$

This result can be shifted to the identity transformation by multiplying on the left by H^{-1} to define an equivalent relative infinitesimal motion:

$$\begin{pmatrix} \omega \\ \mathbf{v} \end{pmatrix} = \mathcal{J}(\mathbf{q})\dot{\mathbf{q}} \quad \text{where} \quad \mathcal{J}(\mathbf{q}) = \left[\left(H^{-1}\frac{\partial H}{\partial q_1} \right)^{\vee}, \cdots, \left(H^{-1}\frac{\partial H}{\partial q_6} \right)^{\vee} \right]. \tag{14}$$

Here

$$\mathbf{v} = R^T \dot{\mathbf{b}}.$$

When the rotations are parameterized as $R = R(q_1,q_2,q_3)$ and the translations are parameterized using Cartesian coordinates $\mathbf{b}(q_4,q_5,q_6) = [q_4,q_5,q_6]^T$, one finds that

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & R^T \end{pmatrix} \tag{15}$$

where J is the Jacobian for the case of rotation.

Differential Operators for $SE(3)$

The differential operators $\partial/\partial\tilde{\xi}_i$ for $i = 1,...,6$ acting on functions on $SE(3)$ are calculated much like they were for the case of $SO(3)$.

For small translational (rotational) displacements from the identity along (about) the i^{th} coordinate axis, the homogeneous transforms representing infinitesimal motions look like

$$H_i(\varepsilon) \stackrel{\triangle}{=} \exp(\varepsilon\tilde{E}_i) \approx \mathbb{I}_{4\times4} + \varepsilon\tilde{E}_i$$

where

$$\tilde{E}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{E}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{E}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tilde{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is often convenient to write these in vector form as

$$\mathbf{e}_i = (\tilde{E}_i)^{\vee}$$

Given that elements of $SE(3)$ (viewed as homogeneous transforms) are parameterized as $H = H(\mathbf{q})$, the differential operators take the form

$$\frac{\partial}{\partial\tilde{\xi}_i} f(H) = \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\left[f(H\circ H_i(\varepsilon)) - f(H)\right] = \frac{d f(H\circ(\mathbb{I}+t\tilde{E}_i))}{dt}\Big|_{t=0} \tag{16}$$

Since H and $H_i(\varepsilon)$ are 4×4 matrices, we henceforth drop the "$\circ$" notation since it is understood as matrix multiplication.

In analogy with the $SO(3)$ case, we observe for the case of $\frac{\partial}{\partial\tilde{\xi}_i}$ that

$$H + \varepsilon H\tilde{E}_i = HH_i(\varepsilon) = H + \varepsilon\sum_{j=1}^{6}\frac{\partial H}{\partial q_j}\frac{\partial q_j^{r,i}}{\partial\varepsilon}\Big|_{\varepsilon=0}.$$

We then have that

$$\tilde{E}_i = \sum_{j=1}^{6} H^{-1}\frac{\partial H}{\partial q_j}\frac{\partial q_j^{r,i}}{\partial\varepsilon}\Big|_{\varepsilon=0},$$

or

$$(\tilde{E}_i)^{\vee} = \sum_{j=1}^{6}\left(H^{-1}\frac{\partial H}{\partial q_j}\right)^{\vee}\frac{\partial q_j^{r,i}}{\partial\varepsilon}\Big|_{\varepsilon=0},$$

which is written as $\mathbf{e}_i = \mathcal{J}(\mathbf{q})\frac{d\mathbf{q}^{r,i}}{d\varepsilon}|_{\varepsilon=0}$ where $\mathcal{J}$ is the $SE(3)$ Jacobian defined earlier. This allows us to solve for

$$\frac{d\mathbf{q}^{r,i}}{d\varepsilon}\Big|_{\varepsilon=0} = \mathcal{J}^{-1}\mathbf{e}_i,$$

which is used to calculate

$$\frac{\partial f}{\partial \tilde{\xi}_i} = \sum_{j=1}^{6} \frac{\partial f}{\partial q_j} \frac{\partial q_j^{r,i}}{\partial \varepsilon}|_{\varepsilon=0}.$$

We use the tilde to distinguish between the full motion and rotation operators. For the case when the rotations are parameterized with ZXZ Euler angles α, β, γ, and translations are parameterized in Cartesian coordinates b_1, b_2, b_3, one finds

$$\frac{\partial}{\partial \tilde{\xi}_i} = \begin{cases} \frac{\partial}{\partial \xi_i} & \text{for } i = 1,2,3 \\ (R^T \nabla_{\mathbf{b}})_{i-3} & \text{for } i = 4,5,6 \end{cases} \tag{17}$$

where $\frac{\partial}{\partial \xi_i}$ is defined in Subsection , and $(\nabla_{\mathbf{b}})_i = \partial/\partial b_i$.

Extending Fixman's Approach to Chains of Rigid Bodies

Inverse of the Mass Matrix for a Single Rigid Body

Let q_1, q_2, q_3 and q_4, q_5, q_6 respectively parameterize the translational and rotational parts of a rigid-body motion. The associated mass matrix, $M(\mathbf{q})$ is of the form

$$M(\mathbf{q}) = \begin{pmatrix} mJ_T J_T^T & 0 \\ 0 & J_R^T I_R J_R \end{pmatrix}$$

where $J_T = J_T(q_1, q_2, q_3) = [\partial \mathbf{x}/\partial q_1, \partial \mathbf{x}/\partial q_2, \partial \mathbf{x}/\partial q_3]$ is the Jacobian for translations (or positions in $\mathbb{R}^3$) and $J_R = J(q_4, q_5, q_6)$ is the $SO(3)$ Jacobian. The inverse of the mass matrix for a single rigid body is

$$M^{-1}(\mathbf{q}) = \begin{pmatrix} mJ_T^{-1} J_T^{-T} & 0 \\ 0 & J_R^{-1} I_R^{-1} J_R^{-T} \end{pmatrix} = \mathcal{J}^{-1} I^{-1} \mathcal{J}^{-T}$$

where $I = (m\mathbb{I}) \oplus I$ is the 6×6 inertia matrix.

The inverse of the mass matrix can be rewritten using the derivatives defined in the previous section. In particular, if we define the $SE(3)$ gradient of a function to be

$$\frac{\partial f}{\partial \tilde{\xi}} = \left[\frac{\partial f}{\partial \tilde{\xi}_1}, \cdots, \frac{\partial f}{\partial \tilde{\xi}_6} \right],$$

then we can apply this gradient to the parameters $\mathbf{q} = [q_1, ..., q_6]^T$ used to parameterize a motion, and find that

$$\frac{\partial \mathbf{q}}{\partial \tilde{\xi}} = \mathcal{J}^{-1}. \tag{18}$$

This means that the inverse of the mass matrix can be written in the Fixman-like form:

$$M^{-1}(\mathbf{q}) = \left[\frac{\partial \mathbf{q}}{\partial \tilde{\xi}} \right] I^{-1} \left[\frac{\partial \mathbf{q}}{\partial \tilde{\xi}} \right]^T.$$

Inverse of the Mass Matrix for a Chain of Rigid Bodies

For a collection of n rigid bodies, the configuration space is

$$(SE(3))^n = SE(3) \times SE(3) \times \cdots \times SE(3)$$

Each rigid body has six degrees of freedom that are described by twists, the i^{th} of which is $\tilde{\xi}_i \in \mathbb{R}^6$, and can be described alternatively by the six parameters $\mathbf{q}_i \in \mathbb{R}^6$. Composite vectors $\tilde{\xi} = [\tilde{\xi}_1^T, ..., \tilde{\xi}_n^T]^T$ and $\mathbf{q} = [\mathbf{q}_1^T, ..., \mathbf{q}_n^T]^T$ can be formed. The inverse of the unconstrained mass matrix for this collection of rigid bodies is then of the form

$$M^{-1}(\mathbf{q}) = \left[\frac{\partial \mathbf{q}}{\partial \tilde{\xi}} \right] [I_1^{-1} \oplus \cdots \oplus I_n^{-1}] \left[\frac{\partial \mathbf{q}}{\partial \tilde{\xi}} \right]^T. \tag{19}$$

Everything then follows using the extension of Fixman's theorem as in the point-mass case, with (19) replacing (3). The same partitioning into soft and hard variables and the same $O(n)$ performance results.

Conclusions

More than 30 years ago, a method for O(n) computation of the determinant of the mass matrix for a chain of point masses was developed by Prof. Marshall Fixman. Whereas this theorem apparently has remained unknown to the multibody and robotics literature, we have applied it to develop O(n) forward dynamics algorithms in a series of papers. The specific contribution of this paper is to extend Fixman's theorem to the case of serial chains of rigid bodies. The associated mathematics required for this extension have also been presented.

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