LARGE KINEMATIC ERROR PROPAGATION IN REVOLUTE MANIPULATORS

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Abstract Understanding how errors propagate in serial revolute manipulators is important for developing better designs and planning algorithms, as well as understanding the practical limitations on accuracy of multi-link arms. In this paper we provide a systematic propagation methodology and numerical example that illustrates how large kinematic errors propagate by convolution on the Euclidean motion group.

Keywords: kinematic error, error propagation, revolute manipulators

1. Introduction

Kinematic errors arising from spatial uncertainties put strong limitations on the performance of serial manipulators. The accumulation of these errors could lead to the failure of executing nominal tasks. Evaluating the propagation effects of kinematic errors is essential in manipulator design, failure prediction, and algorithms planning. It is also important for understanding the practical limitations on accuracy of multi-link arms.

In this paper, we presents a systematic methodology of propagating large errors in revolute manipulators from the point view of Euclidean motion group. Our approach is to treat errors using probability densities on the Euclidean group. Whereas concepts such as integration and convolution of these densities follow in a natural way when considering the Lie group setting [Chirikjian and Kyatkin, 2001], standard concepts associated with the Gaussian distribution in $I\!R^N$ do not follow in a natural way to Lie groups. Several of the most closely related works are reviewed below. These include the theory of Lie groups, robot kinematics, methods for describing spatial uncertainty, and state estimation.

Murray, Li and Sastry [Murray, Li and Sastry, 1994] and Selig [Selig, 1996] presented Lie-group-theoretic notation and terminology to the robotics community, which has now become standard vocabulary. Park and Brockett [Park and Brockett, 1994] showed how dexterity measures can be viewed in a Lie group setting, and how this coordinate-free approach can be used in robot design. Wang and Chirikjian [Wang and Chirikjian, 2004] showed that the workspace densities of manipulators with many degrees of freedom can be generated by solving a diffusion equation on the Euclidean group. Blackmore and Leu [Blackmore and Leu, 1992] showed that problems in manufacturing associated with swept volumes can be cast within a Lie group setting. Kyatkin and Chirikjian [Chirikjian and Kyatkin, 2001] showed that many problems in robot kinematics and motion planning can be formulated as the convolution of functions on the Euclidean group.

Starting with the pioneering work of Brockett [Brockett, 1972], the controls community has embraced group-theoretic problems for many years. This includes PD control on the Euclidean group [Bullo and Murray, 1999; Leonard and Krishnaprasad 1995], tracking problems [Han and Park, 2001; Han, 2004], and estimation [Lo and Eshleman, 1979]. The representation and estimation of spatial uncertainty has also received attention in the robotics and vision literature [Smith and Cheeseman, 1986; Su and Lee, 1992]. Recent work on error propagation described by the concatenation of random variables on groups has also found promising applications in mobile robot navigation [Smith, Drummond, and Roussopoulos, 2003]. We note that while all of these works focus on small errors, our emphasis is a formulation that applies to large errors as well.

2. Review of Rigid-Body Motions

2.1 Euclidean Motion Group

The Euclidean motion group, $SE(N)$, is the semi direct product of $I\!R^N$ with the special orthogonal group, $SO(N)$. We denote elements of $SE(N)$ as $g = (\mathbf{a}, A) \in SE(N)$ where $A \in SO(N)$ and $\mathbf{a} \in I\!R^N$. For any $g = (\mathbf{a}, A)$ and $h = (\mathbf{r}, R) \in SE(N)$, the group law is written as $g \circ h = (\mathbf{a} + A\mathbf{r}, AR)$, and $g^{-1} = (-A^T\mathbf{a}, A^T)$. It is often convenient to express an element of $SE(N)$ as an $(N + 1) \times (N + 1)$ homogeneous transformation matrix of the form:

$$g = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{0}^T & 1 \end{pmatrix}.$$

In this way, rotation and translation are combined into a single matrix. A homogeneous transformation matrix takes the place of the pair $(\mathbf{a}, A)$, and the group operation becomes the matrix multiplication

For example, each element of $SE(2)$ parameterized using polar coordinates can be written as:

$$g(r, \phi, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & r\cos\phi \\ \sin\theta & \cos\theta & r\sin\phi \\ 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where $0 \le \phi, \theta \le 2\pi$ and $0 \le r \le \infty$. $SE(2)$ is a 3-dimensional manifold much like $I\!R^3$. We can integrate over $SE(2)$ using the volume element $d(g(r, \theta, \phi)) = r dr d\theta d\phi$.

2.2 Motion-Group Fourier Transform

The Fourier transform of a function on $G = SE(N)$ is defined as:

$$\hat{f}(p) = \int_G f(g) U(g^{-1}, p)\, d(g) \tag{2}$$

where $d(g)$ is a volume element at g, and $U(g, p)$ is an infinite-dimensional unitary matrix called an irreducible unitary representation, or IUR [Chirikjian and Kyatkin, 2001]. It possess the important homomorphism property, $U(g_1 \circ g_2, p) = U(g_1, p)U(g_2, p)$. One can show that the generalization of the classical Fourier transform in (2) admits a convolution theorem due to the homomorphism property of $U(g, p)$, and that the following inverse transform can be used to reconstruct the original function:

$$f(g) = \int_0^\infty \text{trace}(\hat{f}(p) U(g, p)) p^{N-1} dp. \tag{3}$$

This is because the matrix elements of the full set of IURs form an orthonormal basis with which to expand functions on $SE(N)$.

A number of works [Chirikjian and Kyatkin, 2001] have shown that the matrix elements of the IURs for $SE(2)$ can be expressed as

$$u_{mn}(g(r, \phi, \theta), p) = i^{n-m} e^{-i[n\theta + (m-n)\phi]} J_{n-m}(p\,r) \tag{4}$$

where $J_\nu(x)$ is the ν^{th} order Bessel function, and m and n take values in the integers.

The Fourier inverse transform can be written in terms of elements as

$$f(g) = \sum_{m,n\in \mathbf{Z}} \int_0^\infty \hat{f}_{mn} u_{nm}(g,p) p dp. \quad (5)$$

The motion-group Fourier transform has the property that when applied to convolutions of the form

$$(f_1 * f_2)(g) = \int_G f_1(h) f_2(h^{-1} \circ g) d(h),$$

the result is the product of Fourier transform matrices: $\hat{f}_2(p)\hat{f}_1(p)$.

3. Propagation of Error in Serial Linkages

Suppose we are given a manipulator consisting of two concatenated serial links connected with a revolute joint. One unit is stacked on top of the other one. The proximal unit will be able to reach each frame $h_1 \in SE(3)$ with some error when its proximal end is located at the identity $e \in SE(3)$. This error may be different for each different frame h_1. This is expressed mathematically as a real-valued function of $g_1 \in SE(3)$ which has a peak in the neighborhood of h_1 and decays rapidly away from h_1. If the unit could reach h_1 exactly, this function would be a delta function. Explicitly the error may be described by one of many possible density functions depending on what error model is used. However, it will always be the case that it is of the form $\rho_1(h_1, g_1)$ for $h_1, g_1 \in SE(3)$. That is, the error will be a function of $g_1 \in SE(3)$ for each frame h_1 that the top of the module tries to attain relative to its base. Likewise, the second module will have an error function $\rho_2(h_2, g_2)$ for $h_2, g_2 \in SE(3)$ that describes the distribution of frames around h_2 that might be reached when h_2 is the expected end frame for module 2 relative to its base, and the base of module 2 is located at the identity $e \in SE(3)$.

The error distribution that results from the concatenation of two modules with errors $\rho_1(\cdot)$ and $\rho_2(\cdot)$ results from sweeping the error distribution of the second module by that of the first. This is written mathematically as:

$$\begin{aligned} & \rho(h_1 \circ h_2, g) \\ = \; & (\rho_1 \otimes \rho_2)(h_1 \circ h_2, g) \\ \triangleq \; & \textstyle\int_{SE(3)} \rho_1(h_1, g_1) \rho_2(h_2, g_1^{-1} \circ g) d(g_1). \end{aligned} \quad (6)$$

Here $d(g)$ is the unique bi-invariant integration measure for $SE(3)$ evaluated at g [Chirikjian and Kyatkin, 2001]. Sometimes this is simply written as dg. In the case of no error, the multiplication of homogeneous transforms h_1 and h_2 as $h_1 \circ h_2$ represents the composite change in position and orientation from the base of the lower unit to the interface between units, and from the interface to the top of the upper unit. In the case of inexact kinematics, the error function for the upper unit is shifted by the lower unit $(\rho_2(h_2, g_1^{-1} \circ g))$, weighted by the error distribution of the lower unit $(\rho_1(h_1, g_1))$ and integrated over the support of the error distribution of the lower unit (which is the same as integrating over all of $SE(3)$ since outside of the support of the error distribution the integral is zero). The result of this integration is by definition the error density function around the frame $h_1 \circ h_2$, and this is denoted as $(\rho_1 \otimes \rho_2)(h_1 \circ h_2, g)$. It should be noted that (6) holds regardless of the size of the errors or the form of the error density.

It is often convenient to suppress the explicit dependence of $\rho_i(\cot)$ on h_i, which can be viewed as a constant set of parameters. When this is done, Eq. 6 reduces to a convolution on $SE(N)$.

4. Numerical Example

Consider the three-link planar revolute manipulator shown in Figure 1. Each rigid link has length L, and each joint has some backlash that is described by a probability distribution $f(\theta - \theta_0)$ centered around the value $\theta_0 = 30$ degrees. The error density for a single link is then of the form

$$\rho(g(r, \phi, \theta)) = f(\theta - \theta_0)\delta(\phi - \theta)\delta(r - L)/r \tag{7}$$

where $\delta(\cdot)$ is the usual Dirac delta function in one dimension and an arbitrary element of $g(r, \phi, \theta) \in SE(2)$ is parametrized as in Eq. 1. Integration over $G = SE(2)$ is then integration over all values of $r \in \mathbb{R}^+$ and $\phi, \theta \in [0, 2\pi]$.

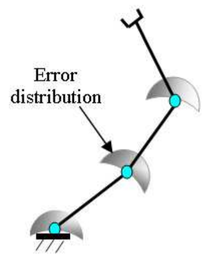

Figure 1. A three-link planar manipulator with joint backlash

In Equation 7, the delta functions enforce the rigidity of the links, and division by r is due to the r in the volume element. The function

$f(\theta)$ has its mode at 0, but the backlashes can be potentially large (i.e., not highly concentrated). For this reason, we cannot take it to be a Gaussian, but rather, a folded Gaussian of the form:

$$\begin{aligned} f(\theta, \sigma) &= \tfrac{1}{2\pi} \textstyle\sum_{n=-\infty}^{\infty} e^{-n^2\sigma^2} e^{in\theta} \\ &= \tfrac{1}{\sqrt{2\pi}\sigma} \textstyle\sum_{n=-\infty}^{\infty} e^{-(\theta-2\pi n)^2/2\sigma^2} \end{aligned} \tag{8}$$

The error density that accumulates at the end effector due to backlashes in each of the joints is computed as the convolution

$$\rho_{ee}(g) = (\rho * \rho * \rho)(g).$$

Computing this numerically by the definition of convolution is not as convenient as using the $SE(2)$-convolution theorem and the corresponding concept of Fourier transform, which is what we shall do here.

Computing the $SE(2)$-Fourier transform of the one-link backlash-error density in Equation 7, one finds (after the delta functions kill the integrals over r and ϕ) that:

$$\begin{aligned} \hat{\rho}_{mn}(p) &= i^{n-m} J_{m-n}(pL) \textstyle\int_0^{2\pi} f(\theta - \theta_0) e^{in\theta} d\theta \\ &= i^{n-m} J_{m-n}(pL) e^{in\theta_0 - n^2\sigma^2}. \end{aligned} \tag{9}$$

Using the convolution theorem, we compute $\hat{\rho}_{ee}(p) = \hat{\rho}\,\hat{\rho}\,\hat{\rho}$, where the matrix elements of $\hat{\rho}$ are given by Equation 9. Then the original error density $\rho_{ee}(g(r, \phi, \theta))$ can be reconstructed by applying the Fourier inversion formula (3) to $\hat{\rho}_{ee}(p)$. Since it is difficult to view the error density $\rho(g)$ graphically, the marginal density function $\rho(r, \phi)$ is computed. The marginal density function $\rho(r, \phi)$ is found by just integrating the Fourier reconstruction formula (3) for $\rho(g)$ with respect to θ as

$$\begin{aligned} \rho(r, \phi) &= \frac{1}{2\pi} \int_0^{2\pi} \rho(g) d\theta \\ &= \sum_{n \in \mathbf{Z}} i^{-n} e^{-in\phi} \int_0^{\infty} \hat{f}_{0n}(p) J_{-n}(pr) p dp. \end{aligned}$$

To validate the results obtained from our convolution-based error propagation method, the error distribution (8) is sampled and applied to each joint of the manipulator directly. Then brute force enumeration is used to obtain the error distribution directly.

The marginal error densities $\rho(r, \phi)$ obtained from both methods are plotted in Figures 2 and 3 with the left one from the propagation method and the right one from brute force. The variance σ^2 of 0.1 is given in Figure 2 and σ^2 of 0.3 in Figure 3.

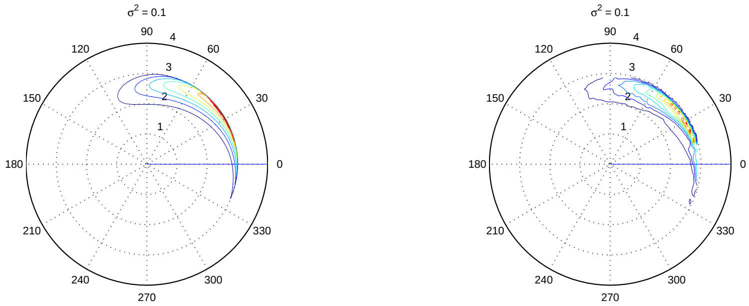

Figure 2. The marginal error density $\rho(r, \phi)$ for σ^2=0.1

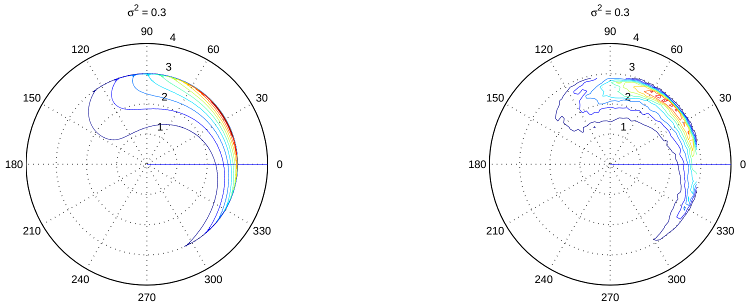

Figure 3. The marginal error density $\rho(r, \phi)$ for σ^2=0.3

For the above computations, the link length L is taken as 1, and 60 samples points are generated for the distribution (8). The infinite-dimensional matrix function $U(g, p)$ in the $SE(2)$ Fourier transform is truncated at finite values of $|m|, |n| \leq 10$ (i.e., the dimension of $U(g, p)$ is $2 \times 10 + 1$). The band-limited approximation still gives very accurate results because the magnitude of the Fourier transform of a sufficiently smooth function can be ignored beyond a certain cutoff frequency. The frequency parameter p is sampled in the interval of 300 with an integration step of 0.2. All the calculations in this example (excluding the brute force method) took less then 3 minutes using Matlab with a 1.0 GHz, 516 MB RAM computer.

5. Conclusions

In this paper it is shown how the accumulation of large kinematic errors in serial manipulators can be computed by performing convolutions on the Euclidean motion group. This theory is demonstrated with the example of a planar revolute manipulator with three links.

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