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Propagation of Errors in Hybrid Manipulators

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Abstract— **Error propagation in hybrid manipulators is addressed here within a rigorous mathematical framework. Understanding how errors propagate in serial manipulators and cascades of platform manipulators is important for developing better designs. In this paper we show that errors propagate by convolution on the Euclidean motion group, $SE(3)$. When local errors are small, they can be described well as distributions on the Lie algebra $se(3)$. We show how the concept of a highly concentrated Gaussian distribution on $SE(3)$ is equivalent to one on $se(3)$. Numerical examples illustrate that convolution and covariance propagation provide the same answers for small errors.**

I. Introduction

In this paper we address how errors propagate on the Euclidean motion group, and specifically examine the accumulation of errors in serial linkages and cascades of parallel platforms. Our approach is to treat errors using probability densities on the Euclidean group.

In the remainder of this section the relevant literature is reviewed, and an overview of rigid-body motions is provided. In Section II the motivating application of error accumulation in serial (and hybrid serial-parallel) manipulators is discussed. In Section III the concept of highly concentrated Gaussian distributions is discussed and several theorems that state their properties are proved. In Section IV closed-form expressions for the convolution of these densities are derived. Section V illustrates this closure with a numerical example. Section VI presents our conclusions and discusses other potential applications of this formulation.

A. Literature Review

Several distinct research fields relate to the work presented in this paper. These include the theory of Lie groups, robot kinematics, and methods for describing spatial uncertainty. We review several of the most closely related works in each of these areas here.

Murray, Li and Sastry [9], McCarthy [8] and Selig [12] presented Lie-group-theoretic notation and terminology to the robotics community, which has now become standard vocabulary. Park and Brockett [11] showed how dexterity measures can be viewed in a Lie group setting, and how this coordinate-free approach can be used in robot design. Wang and Chirikjian [19] showed that the workspace densities of manipulators with many degrees of freedom can be generated by solving a diffusion equation on the Euclidean group. Blackmore and Leu [1] showed that problems in manufacturing associated with swept volumes can be cast within a Lie group setting. Kyatkin and Chirikjian [2], [6] showed that many problems in robot kinematics and motion planning can be formulated as the convolution of functions on the Euclidean group. The representation and estimation of spatial uncertainty has also received attention in the robotics and vision literature. Two classic works in this area are due to Smith and Cheeseman [14] and Su and Lee [15]. Recent work on error propagation by Smith, Drummond and Roussopoulos [13] describes the concatenation of random variables on groups and applies this formalism to mobile robot navigation.

B. Review of Rigid-Body Motions

The Euclidean motion group, $SE(3)$, is the semi direct product of $\mathbb{R}^3$ with the special orthogonal group, $SO(3)$. We denote elements of $SE(3)$ as $g = (\mathbf{a}, A) \in SE(3)$ where $A \in SO(3)$ and $\mathbf{a} \in \mathbb{R}^3$. For any $g = (\mathbf{a}, A)$ and $h = (\mathbf{r}, R) \in SE(3)$, the group law is written as $g \circ h = (\mathbf{a} + A\mathbf{r}, AR)$, and $g^{-1} = (-A^T\mathbf{a}, A^T)$. Alternately, one may represent any element of $SE(3)$ as a 4×4 homogeneous transformation matrix of the form

$$H(g) = \begin{pmatrix} A & \mathbf{a} \\ \mathbf{0}^T & 1 \end{pmatrix},$$

in which case the group law is matrix multiplication.

For small translational (rotational) displacements from the identity along (about) the i^{th} coordinate axis, the homogeneous transforms representing infinitesimal motions look like

$$H_i(\epsilon) \stackrel{\triangle}{=} \exp(\epsilon \tilde{E}_i) \approx \mathbb{I}_{4\times4} + \epsilon \tilde{E}_i$$

where

$$\tilde{E}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{E}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{E}_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$\tilde{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In fact, it can be shown that elements of $SE(3)$ can be described with the exponential parametrization

$$g = g(x_1, x_2, ..., x_6) = \exp\left(\sum_{i=1}^{6} x_i \tilde{E}_i\right). \tag{1}$$

One defines the 'vee' operator, $\vee$, such that

$$\left(\sum_{i=1}^{6} x_i \tilde{E}_i\right)^{\vee} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

The total vector, $\mathbf{x} \in I\!R^6$, can be obtained from $g \in SE(3)$ from the formula

$$\mathbf{x} = (\log g)^{\vee}. \tag{2}$$

II. Propagation of Finite Errors in Serial Linkages

Intuitively, if two rigid parts are manufactured with errors, and those parts are bolted together at an interface, the errors will 'add' in someway. Likewise, a manipulator that is constructed from several subunits, each with some manufacturing error and/or backlash, will have errors that accumulate as the length from base to end effector is traversed. In this section we quantify how errors accumulate in serial and hybrid serial-parallel devices. We formulate this as a convolution of highly concentrated error densities on $SE(3)$.

Suppose we are given a manipulator consisting of two subunits. These units could be Stewart-Gough platforms or serial links connected with revolute joints. One unit is stacked on top of the other one. The proximal unit will be able to reach each frame $h_1 \in SE(3)$ with some error when its proximal end is located at the identity $e \in SE(3)$. This error may be different for each different frame h_1. This is expressed mathematically as a real-valued function of $g_1 \in SE(3)$ which has a peak in the neighborhood of h_1 and decays rapidly away from h_1. If the unit could reach h_1 exactly, this function would be a delta function. Explicitly the error density function may have many forms depending on what error model is used. However, it will always be the case that it is of the form $\rho_1(h_1, g_1)$ for $h_1, g_1 \in SE(3)$. That is, the error will be a function of $g_1 \in SE(3)$ for each frame h_1 that the top of the module tries to attain relative to its base. Likewise, the second module will have an error function $\rho_2(h_2, g_2)$ for $h_2, g_2 \in SE(3)$ that describes the distribution of frames around h_2 that might be reached when h_2 is the expected end frame for module 2 relative to its base, and the base of module 2 is located at the identity $e \in SE(3)$.

The error distribution that results from the concatenation of two modules with errors $\rho_1(\cdot)$ and $\rho_2(\cdot)$ results from sweeping the error distribution of the second module by that of the first. This is written mathematically as:

$$\begin{aligned} & \rho(h_1 \circ h_2, g) \\ = \; & (\rho_1 \otimes \rho_2)(h_1 \circ h_2, g) \\ \triangleq \; & \int_{SE(3)} \rho_1(h_1, g_1)\rho_2(h_2, g_1^{-1} \circ g)d(g_1). \end{aligned} \tag{3}$$

Here $d(g)$ is the unique bi-invariant integration measure for $SE(3)$ evaluated at g [2]. Sometimes this is simply written as dg. In the case of no error, the multiplication of homogeneous transforms h_1 and h_2 as $h_1 \circ h_2$ represents the composite change in position and orientation from the base of the lower unit to the interface between units, and from the interface to the top of the upper unit. In the case of inexact kinematics, the error function for the upper unit is shifted by the lower unit ($\rho_2(h_2, g_1^{-1} \circ g)$), weighted by the error distribution of the lower unit ($\rho_1(h_1, g_1)$) and integrated over the support of the error distribution of the lower unit (which is the same as integrating over all of $SE(3)$ since outside of the support of the error distribution the integral is zero). The result of this integration is by definition the error density function around the frame $h_1 \circ h_2$, and this is denoted as $(\rho_1 \otimes \rho_2)(h_1 \circ h_2, g)$. We illustrated (3) in Figure 1. Parametric distributions that can be used for this application are discussed in Section III, but it should be noted that (3) holds regardless of the size of the errors or the form of the error density.

III. Special Features of Concentrated Probability Functions

Errors in manufactured parts and in the assembly of those parts into larger structures are typically small, but not so small as to be ignored. Therefore, having a way to describe small errors using concentrated probability density functions is useful. This section focuses on the properties of concentrated pdfs on the Euclidean group.

A. Probability Densities Concentrated at the Identity

Suppose instead of a deterministic and exactly measured frame of reference $h \in SE(3)$, we instead have a distribution (or cloud) of frames of reference that are tightly clustered around h. How do we describe such things in a quantitative way? Let us first consider a cloud clustered closely around the identity $e \in SE(3)$. Such a distribution will have most of

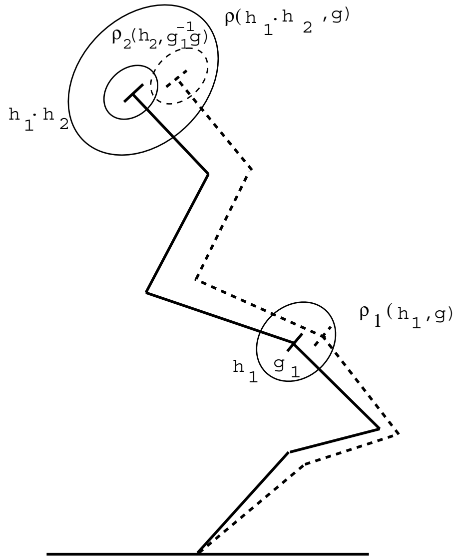

Fig. 1. Error Propagation in Serial Linkages

its mass contained within a ball of radius $\epsilon << 1$ centered at the identity. Here the radius is measured with respect to an appropriate metric, such as can be found in [3]–[5], [7], [10].

A distribution that is concentrated in this way will essentially be a pdf on the Lie algebra $se(3)$, and we therefore can use any number of parametric distributions that are used in $I\!R^N$. For example, the Gaussian distribution

$$\rho(g) = c \exp\left(-\frac{1}{2}\mathbf{x}^T C \mathbf{x}\right) \tag{4}$$

can be used, where $g = g(x_1, x_2, ..., x_6)$ as in Equation 1 and $\mathbf{x}$ is defined as in Equation 2.

We note that while the exponential mapping from $se(3)$ to $SE(3)$ is not bijective, this is irrelevant for two reasons: (1) the set of measure zero for which bijectivitity fails has no effect on nonpathological probability density functions; (2) the small errors to which this mapping is applied are not located at the singularities of the mapping, which are far from the identity.

The normalization constant c is determined by setting

$$\int_G \rho(g) dg = 1$$

so as to make $\rho(g)$ a probability density function. Here dg is the unique bi-invariant integration measure for $SE(3)$. In exponential parameters,

$$dg = w(x_1, x_2, ..., x_6) dx_1 dx_2 \cdots dx_6.$$

Near the identity $w \approx 1$. Therefore, when $\rho(g)$ is tightly concentrated around the identity,

$$\int_G \rho(g) dg \approx \int_{I\!R^6} \rho(g(x_1, x_2, ..., x_6)) dx_1 dx_2 \cdots dx_6.$$

This is true for exponential coordinates and a distribution highly concentrated at the identity. Therefore, the constant c in Equation 4 can be set in the usual way that it is for Gaussian distributions. In particular, if Σ is the matrix of covariances with elements defined by

$$\sigma_{ij} = \int_{I\!R^6} x_i x_j \rho(g(x_1, x_2, ..., x_6)) dx_1 dx_2 \cdots dx_6, \tag{5}$$

then

$$C = \Sigma^{-1} \quad \text{and} \quad c = \left(8\pi^3 |\det\Sigma|^{\frac{1}{2}}\right)^{-1}.$$

Given two probability densities on $SE(3)$, their convolution is defined as

$$(\rho_1 * \rho_2)(g) = \int_G \rho_1(h) \rho_2(h^{-1} \circ g) dh. \tag{6}$$

This can be considered as a special case of (3) when the dependence on h_1 and h_2 either does not exist or is suppressed for notational convenience. If ρ_1 describes a distribution of frames of reference $\{h_1, ..., h_n\}$, and ρ_2 describes a distribution of frames of reference $\{g_1, ..., g_m\}$, then the convolution $(\rho_1 * \rho_2)$ is the distribution that describes the distribution of all pairs $\{h_i \circ g_j | (i,j) \in [1, 2, ..., n] \times [1, 2, ..., m]\}$. In general, since $h_i \circ g_j \neq g_j \circ h_i$, it follows that $(\rho_1 * \rho_2)(g) \neq (\rho_2 * \rho_1)(g)$. However convolutions of two distributions centered tightly around the identity do commute.

In what follows, the functions $\rho_i(g)$ are interpreted as functions with argument in $SE(3)$ described as 4×4 homogeneous transformations. These functions can be extended to have argument in $I\!R^{4\times4}$ in a number of ways, e.g., by setting $\rho(k) = 0$ for all $k \in I\!R^{4\times4} - SE(3)$, or by having $\rho(k)$ decay rapidly to zero as the distance between k and G increases. When such extensions are smooth, then expanding ρ in a Taylor series in $I\!R^{4\times4}$ yields

$$\rho(I + X + O(X^2)) = \rho(I + X)(1 + O(\|x\|)), \tag{7}$$

where $\|x\| = (X, X)^{\frac{1}{2}}$ and $O(X^2)$ is defined in a natural way, i.e., $O(X^2)$ is a matrix with entries each of $O((X, X))$ with $(X, X) = \frac{1}{2} tr(XX^T)$.

Eq. 7 is useful in evaluating expressions in the proofs below. Note that equalities that are presented below are true to $O(\epsilon)$ in the sense that $a(g) = b(g)$ denotes $\int_G |a(g) - b(g)| d(g) = O(\epsilon)$. With this we have the following:

THEOREM 1: Convolution of two functions on $SE(3)$, each tightly focuses at the identity, is the same as convolution on $I\!R^6$ using exponential parameters as coordinates.

PROOF: Let $g = \exp\left(\sum_{i=1}^6 x_i \tilde{E}_i\right)$ and $h = \exp\left(\sum_{i=1}^6 \xi_i \tilde{E}_i\right)$. Let $\rho_i(g)$ for $i = 1, 2$ be functions tightly focused at the identity. Then,

$$\rho_1(h) = \rho_1(I + \sum_{i=1}^{6} \xi_i \tilde{E}_i) \stackrel{\triangle}{=} \tilde{\rho}_1(\boldsymbol{\xi}).$$

Let us define $\tilde{\rho}_2(\cdot)$ in an analogous way. Then, using Eq. 7

and retaining zeroth order terms

$$\begin{array}{rl} & \rho_2(h^{-1}\circ g) \\ = & \rho_2\left(\exp(-\sum_{i=1}^{6}\xi_i\tilde{E}_i)\exp(\sum_{i=1}^{6}x_i\tilde{E}_i)\right) \\ = & \rho_2\left((I-\sum_{i=1}^{6}\xi_i\tilde{E}_i)(I+\sum_{i=1}^{6}x_i\tilde{E}_i)\right) \\ = & \rho_2\left(I+\sum_{i=1}^{6}(x_i-\xi_i)\tilde{E}_i\right) \\ = & \tilde{\rho}_2(\mathbf{x}-\boldsymbol{\xi}). \end{array}$$

Then, the convolution (6) can be written in this special case as

$$(\rho_1 * \rho_2)(g) = \int_{I\!R^6}\tilde{\rho}_1(\boldsymbol{\xi})\tilde{\rho}_2(\mathbf{x}-\boldsymbol{\xi})w(\boldsymbol{\xi})d\boldsymbol{\xi}$$

And since $\boldsymbol{\xi}\approx\mathbf{0}$, $w(\boldsymbol{\xi})\approx 1$. Therefore, we can write:

$$(\rho_1 \tilde{*} \rho_2)(\mathbf{x}) = (\tilde{\rho}_1 \star \tilde{\rho}_2)(\mathbf{x}) \quad (8)$$

where $\star$ is the convolution of functions in $I\!R^6$.

B. Convolution of Probability Densities Shifted from the Identity

The issue of how to describe tightly concentrated distributions around a frame of reference h is handled easily by left translating a distribution defined around the identity:

$$\rho_h(g) = L(h)\{\rho(g)\} = \rho(h^{-1}g).$$

Given two shifted functions, $f_a^1(g) = f^1(a^{-1}\circ g)$ and $f_b^2(g) = f^2(b^{-1}\circ g)$, the convolution is

$$\begin{array}{rl} & (f_a^1 * f_b^2)(g) \\ = & \int_G f_a^1(h)f_b^2(h^{-1}\circ g)dh \\ = & \int_G f^1(a^{-1}\circ h)f^2(b^{-1}\circ h^{-1}\circ g)dh. \end{array}$$

If we define the new variable $k = a^{-1}\circ h$, then $h^{-1} = k^{-1}\circ a^{-1}$. Therefore,

$$(f_a^1 * f_b^2)(g) = \int_G f^1(k)f^2(b^{-1}\circ k^{-1}\circ a^{-1}\circ g)dk.$$

If we define q such that $g = a\circ b\circ q$, then

$$(f_a^1 * f_b^2)(a\circ b\circ q) = \int_G f^1(k)f^2(b^{-1}\circ k^{-1}\circ b\circ q)dk.$$

Now, if f^1 and f^2 have supports that are limited to a small neighborhood around the identity, the only values of k that matter will be close to the identity. The inverse of these values of k also will be close to the identity. The automorphism $k' = b^{-1}\circ k^{-1}\circ b$ preserves closeness to the identity. Therefore, the fact that f^2 has support only in a small neighborhood around the identity, and the fact that k' is close to the identity, means that $f^2(k'\circ q)$ forces q to have importance only near the identity.

Since k is close to the identity, $k = I + \sum_{i=1}^{6}k_i\tilde{E}_i$ Then, by definition

$$(k-I)^\vee = \mathbf{k}\in I\!R^6.$$

Likewise, it can be shown that

$$(b^{-1}\circ k^{-1}\circ b - I)^\vee = -Ad_{b^{-1}}\mathbf{k}$$

where Ad_g is defined by the expression $Ad_g\mathbf{k} = (g\left(\sum_{i=1}^{6}k_i\tilde{E}_i\right)g^{-1})^\vee$. See [2], [9] for the explicit form of $Ad(g)$ as a 6×6 matrix.

Since k and q are both close to the identity, an extension of Theorem 1 can be applied to yield:

$$\begin{array}{l} (f_a^1 * f_b^2)(a\circ b\circ q) = \int_{\mathbf{k}\in I\!R^6}\tilde{f}^1(\mathbf{k}) \\ \tilde{f}^2(\mathbf{q}-Ad_{b^{-1}}\mathbf{k})d\mathbf{k}. \end{array} \quad (9)$$

Note that whereas q is close to the identity, g in general will not be since a and b are not. In order to compute $(f_a^1 * f_b^2)(g)$, one must substitute

$$\mathbf{q} = (\log[(a\circ b)^{-1}\circ g])^\vee$$

into the above expression.

Finally, we note that

$$(f_a^1 * f_b^2)(g) \neq (f_b^2 * f_a^1)(g)$$

even though

$$(f^1 * f^2)(g) \approx (f^2 * f^1)(g)$$

IV. Form Closure for Convolution of Concentrated Gaussians on $SE(3)$

Let us assume that f^1 and f^2 are both concentrated $SE(3)$-Gaussian functions of the form in Equation 4. This section computes Equation 9 explicitly in closed form, and establishes how the mean and variance of each of the initial concentrated distributions 'mix' to result in the mean and variance of their convolution.

We note that while form closure under convolution of Gaussian functions as defined in Equation 4 results trivially from Theorem 1, the case when both functions are shifted is more challenging. Fundamental to all of the calculations in this section is the identity [2]:

$$\begin{array}{rl} & \int_{I\!R^N}\exp(-\frac{1}{2}\mathbf{x}^TM\mathbf{x}-\mathbf{m}^T\mathbf{x})d\mathbf{x} \\ = & (2\pi)^{N/2}|\det M|^{-\frac{1}{2}}\exp\left(\frac{1}{2}\mathbf{m}^TM^{-1}\mathbf{m}\right) \end{array} \quad (10)$$

If $f^i(g)$ are taken to be of the form (4) then direct substitution into (9) and use of (10) with $N=6$ produces the result

$$C_{1*2} = C_2 - C_2(Ad_b^TC_1Ad_b + C_2)^{-1}C_2 \quad (11)$$

This provides all that is required to propagate error densities in closed form rather than numerically performing the convolution. Returning to error propagation and Equation 3, the results of this section are directly applicable be observing that one can define

$$\rho_i(h_i,g_i) = c_i(h_i)\cdot$$

$$\exp\left([(\log(h_i^{-1}g_i))^\vee]^TC_i(h_i)(\log(h_i^{-1}g_i))^\vee\right)$$

In other words, each error density is a Gaussian shifted from the identity to h_i and in addition the covariance matrix ($\Sigma_i = C_i^{-1}$) and scalars c_i depend on the amount of shift.

V. Numerical Example

In this section we present an example in which a cascade of two Stewart-Gough platforms, each with small errors in their leg lengths, is analyzed using the covariance propagation method presented earlier.

Consider a hybrid manipulator of two stacked 6-D Stewart platforms shown in Figure 2. For this Stewart platform, the coordinates of the six connection points at the base and the platform are chosen as

$$\begin{pmatrix} 2\sin(2(i-1)\pi/3 \pm \pi/12) \\ 2\cos(2(i-1)\pi/3 \pm \pi/12) \\ 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 2\sin(2(i-1)\pi/3 \pm 11\pi/12) \\ 2\cos(2(i-1)\pi/3 \pm 11\pi/12) \\ 0 \end{pmatrix},$$

for $i = 1, \cdots, 3$, respectively. The configurations of the first and second module are taken as

$$g_1 = \begin{pmatrix} 0.9755 & -0.1010 & 0.1953 & 2 \\ 0.1545 & 0.9469 & -0.2819 & 2 \\ -0.1564 & 0.3052 & 0.9393 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$g_2 = \begin{pmatrix} 0.9799 & -0.0972 & 0.1742 & 3 \\ 0.1552 & 0.9200 & -0.3599 & 2 \\ -0.1253 & 0.3797 & 0.9166 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The orientation parts of g_1 and g_2 are generated using the Z-Y-X Euler angles, i.e., $(\pi/10, \pi/20, \pi/20)$ for g_1 and $(\pi/8, \pi/25, \pi/20)$ for g_2. Obviously, when two such platforms are stacked, the frame of reference at the end, g_{ee}, is then $g_1 \circ g_2$.

With given g_1 and g_2, the six leg lengths of the first module can be easily calculated as

$$\mathbf{L1} = [5.2692,\ 4.9044,\ 5.5869,\ 4.9083,\ 3.9383,\ 5.4231],$$

and those of the second module as

$$\mathbf{L2} = [4.8020,\ 5.7855,\ 4.3951,\ 4.2350,\ 5.4216,\ 4.1635].$$

In order to test the covariance formula derived in this paper, we generated small deviations of their leg lengths from the above ideal values by assuming that each leg length has a uniformly random error of $\pm 1\%$. Therefore, each leg length was sampled at three values: $L_i, 0.99L_i, 1.01Li$. This generates $n = 3^6$ different frames of reference $\{g_i\}$ that are clustered around g. And while this distribution is not Gaussian, as will be seen, the derived covariance propagation method still works reasonably well. Here g_i is obtained using the forward kinematics method developed in [20].

We compute

$$\mathbf{x}_i = (g^{-1} g_i - e)^{\vee}$$

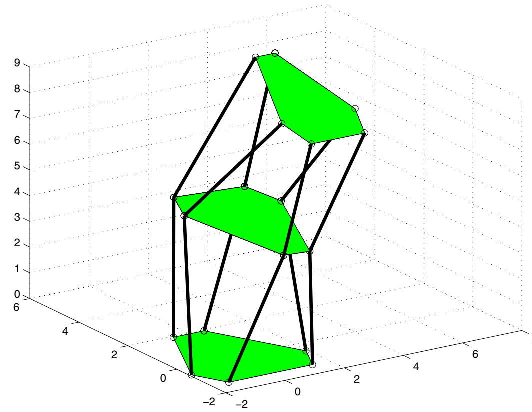

Fig. 2. A hybrid manipulator of two stacked 6-D Stewart-Gough platforms

and then the 'experimental' covariances as

$$\Sigma = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T.$$

and

$$C = \Sigma^{-1}. \quad (12)$$

For leg lengths with $\pm 1\%$ error, the experimental results for the first and second module are computed respectively as

$$C_1 = \begin{pmatrix} 4670.5 & 529.4 & -2380.7 & -340.7 & -245.4 & -180.8 \\ 529.4 & 3238.7 & -2422.6 & -132.0 & -11.0 & 237.4 \\ -2380.7 & -2422.6 & 3333.6 & 419.8 & 435.2 & 324.0 \\ -340.7 & -132.0 & 419.8 & 419.4 & 604.0 & 759.8 \\ -245.4 & -11.0 & 435.2 & 604.0 & 1176.4 & 1401.6 \\ -180.8 & 237.4 & 324.0 & 759.8 & 1401.6 & 1818.9 \end{pmatrix}.$$

and

$$C_2 = \begin{pmatrix} 2014.7 & -419.3 & -1499.4 & -223.8 & 87.4 & 57.6 \\ -419.3 & 3090.1 & -2261.3 & 100.4 & 76.4 & 290.6 \\ -1499.4 & -2261.3 & 4272.1 & 622.3 & 213.3 & 83.0 \\ -223.8 & 100.4 & 622.3 & 1291.9 & 1102.4 & 1120.1 \\ 87.4 & 76.4 & 213.3 & 1102.4 & 1145.3 & 1011.2 \\ 57.6 & 290.6 & 83.0 & 1120.1 & 1011.2 & 1089.5 \end{pmatrix}.$$

Using the covariance propagation formula (11), the covariance of the whole manipulator is obtained:

$$C_{1*2} = \begin{pmatrix} 1259.2 & 57.9 & -984.8 & -5.6 & 200.7 & 107.4 \\ 57.9 & 1520.5 & -1501.4 & -25.6 & 103.4 & 175.3 \\ -984.8 & -1501.4 & 2620.7 & 61.6 & -272.2 & -227.3 \\ -5.6 & -25.6 & 61.6 & 186.2 & 226.9 & 169.8 \\ 200.7 & 103.4 & -272.2 & 226.9 & 405.9 & 260.7 \\ 107.4 & 175.3 & -227.3 & 169.8 & 260.7 & 216.7 \end{pmatrix}.$$

To verify the proposed covariance propagation method, brute force enumeration is also used to get the covariance of the whole manipulator directly. In this case the formula in (12) is used with the n^2 discrete poses obtained by concatenating every element of $\{g_i\}$ with every other, and one obtains:

$$C_{brute} = \begin{pmatrix} 1258.2 & 57.2 & -983.1 & -6.0 & 199.5 & 106.6 \\ 57.2 & 1521.8 & -1499.8 & -29.4 & 96.8 & 170.8 \\ -983.1 & -1499.8 & 2615.4 & 66.3 & -263.4 & -221.4 \\ -6.0 & -29.4 & 66.3 & 176.6 & 212.7 & 159.2 \\ 199.5 & 96.8 & -263.4 & 212.7 & 384.3 & 244.8 \\ 106.6 & 170.8 & -221.4 & 159.2 & 244.8 & 204.9 \end{pmatrix}.$$

As can be seen, these results are in excellent agreement, which serves as a demonstration and validation of the derived formula for the case of small errors. This agreement is quantified in a single number defined using the Hilbert-Schmidt (Frobenius) norm as

$$\epsilon_r = \frac{\|C - C_{brute}\|}{\|C_{brute}\|},$$

where ϵ_r is the error in the C computed by covariance propagation relative to that generated by brute force, and $\|\cdot\|$ denoting the Hilbert-Schmidt (Frobenius) norm. For leg lengths with $\pm 1\%$ error, we found $\epsilon = 0.0113 = 1.13\%$.

Of course, it is of interest to know what happens in the case of other smaller and larger errors, and so we have repeated this experiment with $\pm 0.1\%$, $\pm 0.5\%$, $\pm 0.8\%$, $\pm 2\%$, $\pm 3\%$, and $\pm 5\%$ errors on leg lengths. The trend is graphed in Figure 3. Clearly the approximations used in the derivation of covariance propagation break down as the errors become large.

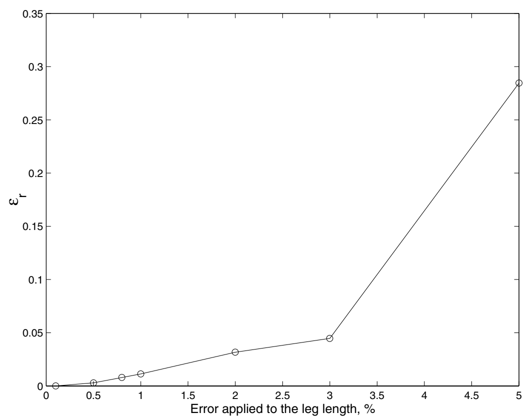

Fig. 3. The accuracy of the proposed propagation covariance method

VI. Conclusions

Quantifying the intuitive notion of how spatial errors 'add' has been addressed in this paper. It was shown that even though the concept of a Gaussian distribution does not completely generalize when considering the case of Lie-group-valued argument, an appropriate concept does exist when considering highly concentrated distributions. This paper worked out the details of how Gaussian distributions are defined in this context, what their properties are, and how they can be applied to compute the propagation of covariances in serial manipulators. Thoerems regarding the properties of these distributions were proven. The computations performed show that such distributions have the desired closure properties in order for them to be useful in error propagation problems in robotics.

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