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DRAFT: POSE CHANGES FROM A DIFFERENT POINT OF VIEW

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ABSTRACT

For more than a century, rigid-body displacements have been viewed as affine transformations described as homogeneous transformation matrices wherein the linear part is a rotation matrix. In group-theoretic terms, this classical description makes rigid-body motions a semi-direct product. The distinction between a rigid-body displacement of Euclidean space and a change in pose from one reference frame to another is usually not articulated well in the literature. Here we show that, remarkably, when changes in pose are viewed from a space-fixed reference frame, the space of pose changes can be endowed with a direct product group structure, which is different than the semi-direct product structure of the space of motions. We then show how this new perspective can be applied more naturally to problems such as monitoring the state of aerial vehicles from the ground, or the cameras in a humanoid robot observing motions of its hands.

1 INTRODUCTION

In elementary courses on manipulator kinematics, students learn that the motion from A to B (as seen in A) can be expressed as a homogeneous transformation, written here as ${}^{A}H_{B}$, and likewise from B to C. They also learn that the concatenation of rigid-body motions when going from frame A to C is then the product of these transformations, ${}^{A}H_{C} = {}^{A}H_{B}\,{}^{B}H_{C}$ and that ${}^{A}H_{B}^{-1} = {}^{B}H_{A}$. In addition, they learn that the spatial transformation describing this relative motion from A to B (as seen in A) is 'the same' as the transformation that converts the coordinates of position vectors seen relative to B back into the coordinates of the same positions as seen in A. This is a natural description of motion for serial kinematic structures, as put forth in [13]. This sequential mechanical view of kinematics has proved its worth over many years, however, it is inherently based on a confounding of the role of coordinates and spatial transformations.

In this paper we consider the question of how to properly model motion between two poses when considered from a third point of view corresponding to an observer. This scenario is depicted in the context of a humanoid robot in Figure 1 in which frames A and B describe the pose of the robot's hand at two different times, and O is the frame attached to the head where the visual sensors of the robot are placed. A similar scenario arises in the ground-based remote control of aerial vehicles, in which case O would be the ground frame. A key contribution of the paper is to show that (at least in many robotics applications) such

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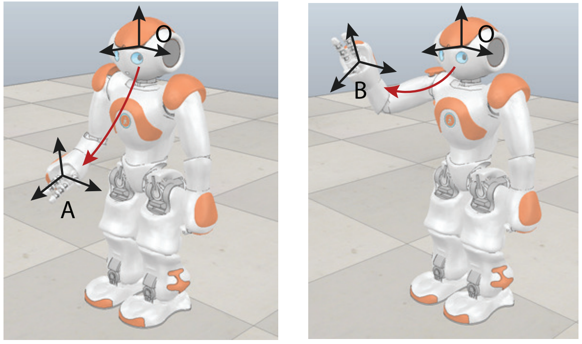

FIGURE 1. Demonstrating the three frame scenario with a separate observer frame O for a Humanoid Robot

motion should not be modelled as a Euclidean motion of space, but rather as a translation and rotation of the frame A (representing an initial pose) to move it to the position and orientation of frame B (representing a final pose) *without moving the whole of space along with the frame*. To make this point, we argue that motion (either a Euclidean motion or a pose change) should be modelled by a group action acting on an appropriate space representing the poses. The group associated with rigid-body or Euclidean motion is the Special Euclidean group SE(3) and it acts on points in Euclidean 3-space. When a frame of reference A is fixed to Euclidean space, then a Euclidean motion of space will carry the frame along with it to a new location and orientation characterised by a second frame B.[1] It is well known that the change in position and orientation from frame A to frame B fully characterizes the Euclidean motion of space, however, it is less well known that its expression in terms of a homogeneous transformation ${}^{A}H_{B}$ is dependent on the observer. In particular, we show that when this Euclidean motion is expressed from the point of view of a separate observer O, the resulting element of the matrix group SE(3) *is not* the same element that would be obtained if the observer is located at the origin of frame A, the classical formulation of Euclidean motion in robotics.

We go on to consider the action of a different group, the *pose change group*, on the space of poses (represented by frames). The pose change group, PCG(3), is the direct product of the Special Orthogonal group SO(3) and the additive translation group $\mathbb{R}^3$. To emphasize the difference to the Special Euclidean group we initially use pairs $(R,\mathbf{t})$ of rotations R and translations $\mathbf{t}$ to describe both groups so that the difference in the group laws becomes immediately apparent. We illustrate how the direct-product group structure articulated here results in geodesics on pose space that can be more natural in some trajectory-generation applications. We believe that this observation has significant implications in modelling, path planning and control for pose change when observed from a position that is not the pose itself.

The remainder of the paper is structured as follows. First, the traditional view of Euclidean motions as elements of the group $SE(3)$ is reviewed. We then review the mathematics of coordinate changes, which is known in kinematics but is more often used in computer vision. We then review elements of screw theory, focusing on the screw parameters of Euclidean motions that are invariant under coordinate changes (or changes in perspective). Then we take a mathematical view to explain the differences between spaces of poses, pose changes, Euclidean motions, and coordinate transformations, all of which are different mathematical objects that get conflated in the robot kinematics literature. The paper concludes with an application of the new view of pose changes to a trajectory generation problem.

2 THE TRADITIONAL VIEW

The literature on kinematics is vast and spans multiple centuries. The earliest, and most fundamental, work on kinematics was formulated in a coordinate-free setting both for pure rotations [7, 15, 28] and full Euclidean motions [5, 8]. Such formulations were put forth in an era in which the computational benefits of coordinates had not yet been realized. But this also endows such works with an elegance that is sometimes lacking in recent work. Modern treatments of kinematics can be found in [1, 3, 6, 17, 19, 24, 25, 27, 30]. A mainstay of robot kinematics is screw theory, which is described in detail in works including [4, 11, 12, 22, 29]. The part of this paper that reviews the group-theoretic and screw-theoretic aspects of Euclidean motions follows that in [9].

The group of rigid-body motions, $SE(3)$, consists of pairs $(R,\mathbf{t})$ where $R \in SO(3)$ is a *rotation*, and $\mathbf{t} \in \mathbb{R}^3$ is a *translation*. The group law is

$$(R_1,\mathbf{t}_1)\circ(R_2,\mathbf{t}_2) = (R_1R_2, R_1\mathbf{t}_2+\mathbf{t}_1). \tag{1}$$

From this law, and the fact that the identity element is $(I,\mathbf{0})$, it follows that the inverse of a rigid-body displacement is

$$(R,\mathbf{t})^{-1} = (R,-R^T\mathbf{t})$$

where T denotes the transpose of a matrix.

Given a point $\mathbf{x}\in\mathbb{R}^n$, an element $(R,\mathbf{t})\in SE(3)$ acts as

$$(R,\mathbf{t})\cdot\mathbf{x} = R\mathbf{x}+\mathbf{t}, \tag{2}$$

and this satisfies the definition of a (left) *group action*. That is,

$$((R_1,\mathbf{t}_1)\circ(R_2,\mathbf{t}_2))\cdot\mathbf{x} = (R_1,\mathbf{t}_1)\cdot((R_2,\mathbf{t}_2)\cdot\mathbf{x})$$

[1] In formal mathematical terms this can be expressed as an action of the group SE(3) on the orthonormal frame bundle $F_O(T\mathbb{E}^3)$ of the tangent bundle $T\mathbb{E}^3$ of Euclidean 3-space $\mathbb{E}^3$.

and

$$(I, \mathbf{0}) \cdot \mathbf{x} = \mathbf{x}.$$

Written slightly more abstractly, if $(G, \circ)$ is a group and X is a set, then $(G, \circ)$ is said to act on X if there is an operation $\cdot$ (called an 'action'), such that for the group identity $e \in G$ and every pair $g_1, g_2 \in G$ we have

$$e \cdot x = x \quad \text{and} \quad (g_1 \circ g_2) \cdot x = g_1 \cdot (g_2 \cdot x) \in X. \tag{3}$$

Homogeneous transformation matrices of the form

$$H(R, \mathbf{t}) = \begin{pmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}. \tag{4}$$

are traditionally used to represent rigid-body displacements [6, 13, 24]. It is easy to see by the rules of matrix multiplication and the composition rule for rigid-body motions that

$$H((R_1, \mathbf{t}_1) \circ (R_2, \mathbf{t}_2)) = H(R_1, \mathbf{t}_1) H(R_2, \mathbf{t}_2).$$

Likewise, the inverse of a homogeneous transformation matrix represents the inverse of a motion:

$$H((R, \mathbf{t})^{-1}) = [H(R, \mathbf{t})]^{-1}.$$

In this notation, vectors in $\mathbb{R}^3$ are augmented by appending a "1" to form a vector

$$X = \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, \tag{5}$$

and this preserves the properties of the group action, computed as a matrix vector product.

3 DIFFERENCES BETWEEN POSE, COORDINATE TRANSFORMATIONS, POSE CHANGES, AND EUCLIDEAN MOTIONS

A *Euclidean motion* is a handedness-preserving isometry of Euclidean space. Such motions can be described using the group-theoretic tools in the previous section. Under the action of a Euclidean motion, all points in Euclidean space move as a single rigid-body. A Euclidean motion that takes frame A to frame B is a well-defined object even without introducing any coordinates or a frame of reference in which to observe the motion.

A *pose* is different than a Euclidean motion in that it describes a position and orientation (or frame) in space. Simply put, if a (right-handed) frame is marked on Euclidean space, the result is a pose. Such a pose (or a set of poses) can exist as a geometric construction without coordinates. The set of all frames without specified coordinates or group operation is also a well-defined mathematical object the so-called *orthonormal frame bundle* $F_O(T\mathbb{E}^3)$ of the tangent bundle $T\mathbb{E}^3$ of Euclidean 3-space $\mathbb{E}^3$. Focussing on right-handed frames only, pose space carries the structure of an SE(3)-*torsor*. It has exactly the same topological and differential properties as $SE(3)$, but without the group structure.

A *change of pose* is a conversion of one pose into another. For example, given a moving finite rigid body with a body-fixed reference frame, the reference frame is initially at one pose at the initial time and then it is at another pose at a subsequent time. We emphasize that this is different than a Euclidean motion where a copy of the whole of Euclidean space attached to the body moves with it and maps back into itself as the body undergoes the motion.

Once a global reference frame is established, then both poses and Euclidean motions can be described using homogeneous matrices. Such matrices can also play a different role as *coordinate transformations* to describe the motion from different perspectives. That is, whereas the phenomena of Euclidean motions and pose changes are well defined regardless of how they are observed, when seeking to express such phenomena numerically, coordinate frames must be drawn both in the object that moves and in a fixed copy of Euclidean space. Coordinate transformations then describe two distinctly different changes in perspective: (1) changes in the space-fixed reference frame in which motions and poses are described; (2) changes in the way reference frames are attached to moving bodies.

The following section describes in precise mathematical terms how coordinate changes affect the homogeneous transformations describing Euclidean motions.

4 CONJUGATION AND CHANGE OF VIEW

The group-theoretic treatment reviewed in the previous sections uses "light" notation that suppresses information about which frames the displacements are viewed from. For example, the Cartesian coordinates of the position of a point in $\mathbf{x} \in \mathbb{R}^n$ depends on which frame the point is viewed from.

To make this point clear we now introduce four frames of reference: O, A, B, and C, and use "heavy" notation. Then ${}^{O}\mathbf{x}$, ${}^{A}\mathbf{x}$, ${}^{B}\mathbf{x}$, ${}^{C}\mathbf{x}$, are the coordinates of $\mathbf{x}$ as seen in O, A, B, and C, respectively.[2] This much is common knowledge in the fields of

[2] In linear algebra terms, this means that we select the origin of the respective frame as the zero point in our coordinates and the axes of the frame as a basis of the vector space describing linear translations of the origin.

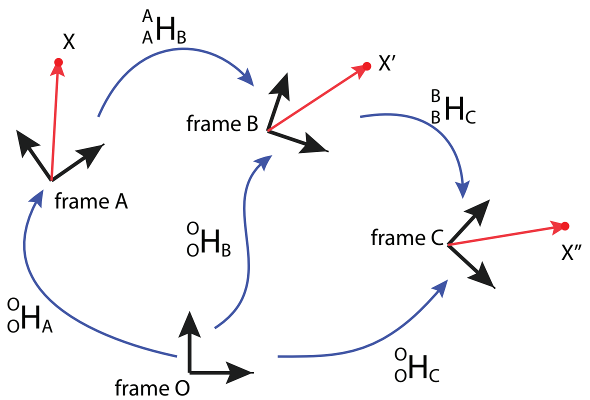

FIGURE 2. Relationship between the frames O, A, B and C

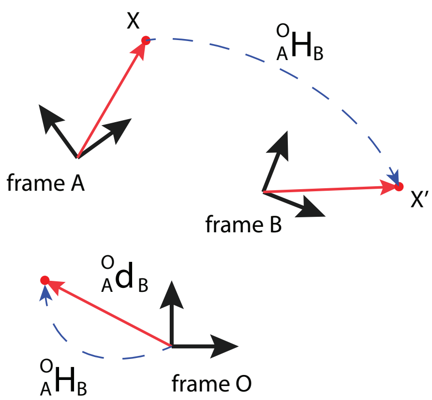

FIGURE 3. The meaning of the vector ${}^{O}_{A}\mathbf{d}_B$

Robotics, Mechanisms, and Computer Vision. What is less commonly known is that a *three-indexed* rigid-body motion of the form

$$ {}^{O}_{A}H_B = H({}^{O}_{A}R_B, {}^{O}_{A}\mathbf{d}_B) \tag{6}$$

is required to avoid ambiguity when considering the rigid-body displacement from A to B as seen in O. Usually in robotics and mechanism kinematics ${}^{A}_{A}H_B$ is considered, which allows for simpler notation such as ${}^{A}H_B$. But to make the fundamental points in this paper, we keep all three indices. Figure 2 illustrates the various reference frames used throughout this paper.

We have used the notation ${}^{O}_{A}\mathbf{d}_B$ in Equation (6) to emphasize the fact that this vector is not the translation vector from the origin of frame A to the origin of frame B but rather the vector describing the displacement of the origin of frame O under the action of the Euclidean motion of space that is represented by ${}^{O}_{A}H_B$. Recall that this is the motion that moves frame A to frame B and refer to Figure 3 for an illustration of that motion.

Within the three frame context, traditional statements about

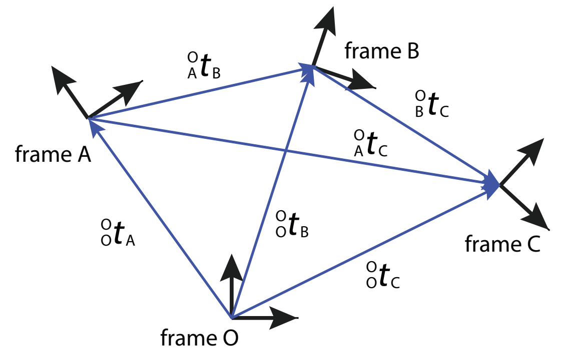

FIGURE 4. Translation Vectors ${}^{O}_{O}\mathbf{t}_A$, ${}^{O}_{O}\mathbf{t}_B$, ${}^{O}_{O}\mathbf{t}_C$, ${}^{O}_{A}\mathbf{t}_B$, ${}^{O}_{B}\mathbf{t}_C$

the concatenation of reference frames such as

$$ {}^{A}H_B\, {}^{B}H_C = {}^{A}H_C$$

are instead written as

$$ {}^{A}_{A}H_B\, {}^{B}_{B}H_C = {}^{A}_{A}H_C. \tag{7}$$

Moreover, from Figure 4 it is clear that translations viewed from the same reference frame can all be related by traditional vector arithmetic. For example,

$$ {}^{O}_{A}\mathbf{t}_B = {}^{O}_{O}\mathbf{t}_B - {}^{O}_{O}\mathbf{t}_A\,,\ {}^{O}_{B}\mathbf{t}_C = {}^{O}_{O}\mathbf{t}_C - {}^{O}_{O}\mathbf{t}_B$$

and

$$ {}^{O}_{A}\mathbf{t}_C = {}^{O}_{A}\mathbf{t}_B + {}^{O}_{B}\mathbf{t}_C = {}^{O}_{O}\mathbf{t}_C - {}^{O}_{O}\mathbf{t}_A. \tag{8}$$

This much is clear simply from classical vector arithmetic.

We reiterate that the vector ${}^{O}_{A}\mathbf{d}_B$ in (6) is in general not equal to the translation vector ${}^{O}_{A}\mathbf{t}_B$ unless $O = A$, but it can always be related to it, see Equation(13) below.

We now examine the relationship between ${}^{O}_{A}H_B$, ${}^{A}_{A}H_B$, and ${}^{O}_{O}H_A$. Let $\mathbf{x}$ be a specific point in Euclidean space and let X be its description in homogeneous coordinates as in (5). The coordinates of this can be described in any reference frame. For example, if ${}^{A}X$ is the homogeneous description of $\mathbf{x}$ as seen in frame A, then

$$ {}^{O}X = {}^{O}\mathcal{H}_A\, {}^{A}X.$$

We have used the calligraphic notation ${}^{O}\mathcal{H}_A$ to emphasize that in this context we are not considering a *Euclidean motion* ${}^{O}H_A$ of space, indeed we are talking about a single fixed point in space $\mathbf{x}$, but a *coordinate transformation* of how this point is represented by the entries of the vector X.

Alternatively, if this point is moved to a new point $\mathbf{x}'$ by the same Euclidean motion that moves frame A to frame B, then

$$^{A}X' = {}^{A}_{A}H_{B}\,{}^{A}X\,. \tag{9}$$

This motion of the point $\mathbf{x}$ to the new point $\mathbf{x}'$ should not be confused with the coordinate change

$$^{A}X' = {}^{A}\mathcal{H}_{B}\,{}^{B}X'\,. \tag{10}$$

The relationship that links these two equations is the fact that the coordinates of the points $\mathbf{x}$ and $\mathbf{x}'$ are the same when viewed in the correct reference frames, that is,

$$^{B}X' = {}^{A}X,$$

and that the matrix equation

$$^{A}\mathcal{H}_{B} = {}^{A}_{A}H_{B} \tag{11}$$

holds, i.e. both objects have the same numerical representation in the matrix group $\mathrm{SE}(3)$ although they have very different conceptual meanings. Indeed, this is where (7) comes from.

Expressions analogous to those above which involve the frame O include

$$^{O}X' = {}^{O}_{A}H_{B}\,{}^{O}X\,,\; {}^{O}X = {}^{O}\mathcal{H}_{A}\,{}^{A}X = {}^{O}_{O}H_{A}\,{}^{A}X\,.$$

Combining the above equations and localizing using the fact that $\mathbf{x}$ is an arbitrary point yields

$$\boxed{{}^{O}_{A}H_{B} = {}^{O}\mathcal{H}_{A}\,{}^{A}_{A}H_{B}\,{}^{O}\mathcal{H}_{A}^{-1} = {}^{O}_{O}H_{A}\,{}^{A}_{A}H_{B}\,{}^{O}_{O}H_{A}^{-1}\,.} \tag{12}$$

Such conjugation/similarity transformations are well-known in kinematics [6,24]. The three index notation used here is inspired by [31]. In the present context, ${}^{A}_{A}H_{B}$ describes a Euclidean motion, whereas ${}^{O}\mathcal{H}_{A}$ is a coordinate transformation whose matrix representation is the same as that of the Euclidean motion ${}^{O}_{O}H_{A}$. The result of the conjugation in (12) is again a Euclidean motion, but now seen with respect to a different coordinate system.

Performing the matrix multiplications in (12) gives rotation and translation parts

$${}^{O}_{A}R_{B} = {}^{O}_{O}R_{A}\,{}^{A}_{A}R_{B}\,{}^{O}_{O}R_{A}^{-1}$$

and

$$\boxed{{}^{O}_{A}\mathbf{d}_{B} = \left(I - {}^{O}_{A}R_{B}\right){}^{O}_{O}\mathbf{t}_{A} + {}^{O}_{A}\mathbf{t}_{B}} \tag{13}$$

where

$${}^{O}_{A}\mathbf{t}_{B} = {}^{O}_{O}R_{A}\,{}^{A}_{A}\mathbf{t}_{B}\,.$$

Note from (13) that

$$\boxed{{}^{O}_{A}\mathbf{d}_{B} \neq {}^{O}_{A}\mathbf{t}_{B}\,,} \tag{14}$$

but given ${}^{A}_{A}H_{B}$ and ${}^{O}_{O}H_{A}$, all the necessary information exists to interconvert between ${}^{O}_{A}\mathbf{t}_{B}$ and ${}^{O}_{O}R_{A}\,{}^{A}_{A}\mathbf{t}_{B}$.

It can also be shown that homogeneous transforms describing concatenated displacements $A \to B \to C$ all observed from frame O can be written as[3]

$${}^{O}_{A}H_{C} = {}^{O}_{B}H_{C}\,{}^{O}_{A}H_{B}. \tag{15}$$

Explicitly,

$${}^{O}_{B}H_{C}\,{}^{O}_{A}H_{B} = \left({}^{O}_{O}H_{B}\,{}^{B}_{B}H_{C}\,{}^{O}_{O}H_{B}^{-1}\right)\left({}^{O}_{O}H_{A}\,{}^{A}_{A}H_{B}\,{}^{O}_{O}H_{A}^{-1}\right).$$

The parentheses are only for emphasis and can be removed. Then using the fact that

$${}^{O}_{O}H_{B}^{-1}\,{}^{O}_{O}H_{A} = {}^{A}_{A}H_{B}^{-1}$$

allows for the simplification

$${}^{O}_{B}H_{C}\,{}^{O}_{A}H_{B} = {}^{O}_{O}H_{B}\,{}^{B}_{B}H_{C}\,{}^{O}_{O}H_{A}^{-1}.$$

But premultiplying by $I = {}^{O}_{O}H_{A}\,{}^{O}_{O}H_{A}^{-1}$ and using the fact that

$${}^{A}_{A}H_{B} = {}^{O}_{O}H_{A}^{-1}\,{}^{O}_{O}H_{B} \tag{16}$$

gives

$${}^{O}_{B}H_{C}\,{}^{O}_{A}H_{B} = {}^{O}_{O}H_{A}\,{}^{A}_{A}H_{B}\,{}^{B}_{B}H_{C}\,{}^{O}_{O}H_{A}^{-1}.$$

[3]The reader is invited to compare the order of terms in Equations (7) and (15). The latter is the usual linear algebra expression where matrix multiplication represents concatenation of affine maps given that all matrix representations are chosen with respect to the same basis. The former has no linear algebra equivalent and is specific to rigid body kinematics.

This, in combination with (7), gives (15).

As a consequence, the rotational parts of (15) follow as

$$ {}^{O}_{A}R_{C} = {}^{O}_{B}R_{C}\,{}^{O}_{A}R_{B}. \tag{17} $$

We note in passing that substituting (16) into (12) gives

$$ {}^{O}_{A}H_{B} = {}^{O}_{O}H_{B}\,{}^{O}_{O}H_{A}^{-1} \tag{18} $$

5 SCREW PARAMETERS

According to Chasles' Theorem [8]: (1) Every motion of a rigid body can be considered as a translation in space and a rotation about a point; (2) Every spatial displacement of a rigid body can be equivalently affected by a single rotation about an axis and translation along the same axis.

This forms the foundation for Screw Theory, as developed in [4, 12, 14].

In screw theory, the axis in space is described by a line

$$ \mathbf{L}(t) = \mathbf{p} + t\mathbf{n}, \qquad \forall \qquad t \in \mathbb{R}, $$

where $\mathbf{p}$ is the unique vector pointing to the screw axis and intersecting it at a right angle. Therefore, $\mathbf{p}\cdot\mathbf{n} = 0$.

Any Euclidean motion viewed as a screw displacement can be expressed in terms of $(\mathbf{n},\mathbf{p})$ and the motion paramters (θ, d), i.e., the angle of rotation and the distance travelled around/along $\mathbf{n}$, is given as

$$ H(\mathbf{n},\mathbf{p},\theta,d) = \begin{pmatrix} e^{\theta N} & (\mathbb{I} - e^{\theta N})\mathbf{p} + d\mathbf{n} \\ \mathbf{0}^{T} & 1 \end{pmatrix}. \tag{19} $$

The screw axis parameters $(\mathbf{n},\mathbf{p})$ and motion parameters (θ, d) can be extracted from a given rigid displacement $(R, \mathbf{d})$. Since this is widely known for pure rotations, half the problem is already solved, i.e., $\mathbf{n}$ and θ are calculated from $R = e^{\theta N}$ by observing that

$$ \mathrm{tr}(R) = 1 + 2\cos\theta \text{ and } R - R^{T} = 2\sin\theta\, N, $$

which can be solved for θ and $N = \hat{\mathbf{n}}$, the skew-symmetric matrix associated with $\mathbf{n}$. Since

$$ (\mathbb{I} - R)\mathbf{p} + d\mathbf{n} = \mathbf{d} \qquad \text{and} \qquad \mathbf{p}\cdot\mathbf{n} = 0, $$

it follows from $\mathbf{n}\cdot e^{\theta N}\mathbf{p} = \mathbf{n}\cdot\mathbf{p} = 0$ that

$$ d = \mathbf{d}\cdot\mathbf{n}. \tag{20} $$

It is known that θ and d are invariant under conjugation by arbitrary motions [9] and so the screw parameters $({}_{A}\theta_{B}, {}_{A}d_{B})$ depend only on the two frames A and B. Interestingly, from (13) and (20) we have

$$ \boxed{{}_{A}d_{B} = {}^{O}_{A}\mathbf{d}_{B}\cdot {}^{O}_{A}\mathbf{n}_{B} = {}^{O}_{A}\mathbf{t}_{B}\cdot {}^{O}_{A}\mathbf{n}_{B}.} \tag{21} $$

Therefore, the screw parameters $({}_{A}\theta_{B}, {}_{A}d_{B})$ that are invariant to the choice of O can be obtained either from ${}^{O}_{A}H_{B} = H({}^{O}_{A}R_{B}, {}^{O}_{A}\mathbf{d}_{B})$ or from $({}^{O}_{A}R_{B}, {}^{O}_{A}\mathbf{t}_{B})$.

6 DESCRIBING POSE CHANGES AS DIRECT PRODUCT OPERATIONS

An altogether different way to describe changes in pose is formulated in this section. Suppose we are sitting at a space-fixed frame O and we are watching frame A move to frame B. We will see the origin of A move to the origin of B. This gives the vector ${}^{O}_{A}\mathbf{t}_{B}$. Again, this is not the same as ${}^{O}_{A}\mathbf{d}_{B}$. Similarly, the rotational part of the displacement as seen from this reference frame will be of the form

$$ {}^{O}_{A}R_{B} = \exp\left({}_{A}\theta_{B}\,{}^{O}_{A}\hat{\mathbf{n}}_{B}\right). $$

Here ${}_{A}\theta_{B}$ does not depend on O because it is one of the conjugation-invariant quantities for $SE(3)$ as discussed in Section 5.

Since ${}^{O}_{A}\mathbf{n}_{B} = {}^{O}_{O}R_{A}\,{}^{A}_{A}\mathbf{n}_{B}$ this gives

$$ {}^{O}_{A}R_{B} = {}^{O}_{O}R_{A}\,{}^{A}_{A}R_{B}\,{}^{O}_{O}R_{A}^{T}, $$

which is just the rotational part of (12).

Given two rotation-translation pairs $\left({}^{O}_{A}R_{B}, {}^{O}_{A}\mathbf{t}_{B}\right)$ and $\left({}^{O}_{B}R_{C}, {}^{O}_{B}\mathbf{t}_{C}\right)$ (where the actual translations ${}^{O}_{A}\mathbf{t}_{B}$ and ${}^{O}_{B}\mathbf{t}_{C}$ are used instead of ${}^{O}_{A}\mathbf{d}_{B}$ and ${}^{O}_{B}\mathbf{d}_{C}$), a simple rule for moving from A to C follows by combining (17) and (8). That is, the combination of

$$ {}^{O}_{A}\mathbf{t}_{C} = {}^{O}_{B}\mathbf{t}_{C} + {}^{O}_{A}\mathbf{t}_{B} $$

and

$$ {}^{O}_{A}R_{C} = {}^{O}_{B}R_{C}\,{}^{O}_{A}R_{B} $$

defines a *direct product* of the rotation and translation groups

$$ \left({}^{O}_{A}R_{C}, {}^{O}_{A}\mathbf{t}_{C}\right) = \left({}^{O}_{B}R_{C}, {}^{O}_{B}\mathbf{t}_{C}\right) \bullet \left({}^{O}_{A}R_{B}, {}^{O}_{A}\mathbf{t}_{B}\right) \tag{22} $$

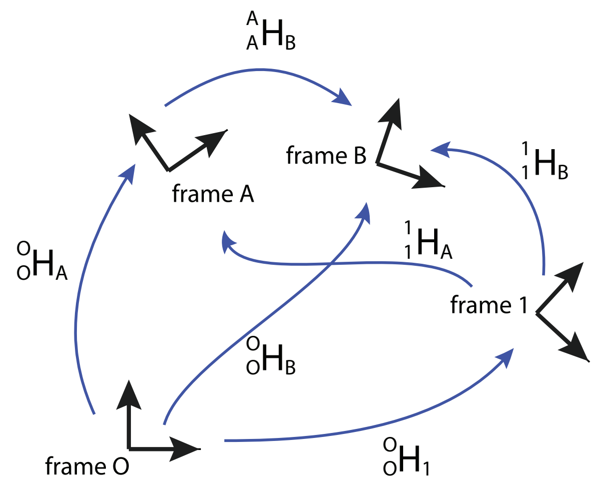

FIGURE 5. Transformation of the Observer Frame from O to 1

This direct product defines the *pose change group*:

$$\mathrm{PCG}(3) = \mathrm{SO}(3) \times \mathbb{R}^3. \tag{23}$$

Let $\mathbf{x}$ denote a point in space, which can be thought of as the origin of a reference frame X. Then, for example, the translation ${}^{O}_{A}\mathbf{t}_X$ is actually what the position of $\mathbf{x}$ is from the origin of A, but as seen from the orientation of O.

6.1 CHANGE OF OBSERVER FRAME

An action of the pose change group $\mathrm{PCG}(3)$ on pose space can be defined as

$$(Q,\xi) \odot (R,\mathbf{t}) \doteq \left(QRQ^T, Q\mathbf{t}\right), \tag{24}$$

where $(Q,\xi) \in \mathrm{PCG}(3)$. Note that the right hand side of (24) does not depend on $\xi \in \mathbb{R}^3$. In other words, the $\mathbb{R}^3$ part of $\mathrm{PCG}(3) = \mathrm{SO}(3) \times \mathbb{R}^3$ acts trivially. The action $\odot$ is associated with a change of observer frame as follows.

If the motion from A to B is viewed from frame 1 rather than frame O, the resulting homogeneous transformation is

$${}^{1}_{A}H_B = {}^{O}_{O}H_1^{-1}\, {}^{O}_{A}H_B\, {}^{O}_{O}H_1\,. \tag{25}$$

Obviously, as a consequence

$${}^{1}_{A}R_B = {}^{O}_{O}R_1^T\, {}^{O}_{A}R_B\, {}^{O}_{O}R_1\,, \tag{26}$$

which is also the rotational part of the pose change parameters resulting by changing the observer from O to 1. Figure 5 shows how this change in the observer frame affects the observed motion.

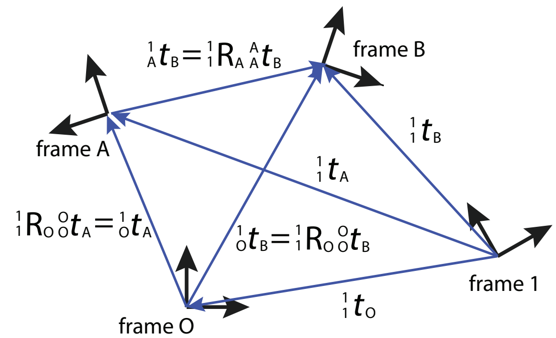

FIGURE 6. Coordinates of All Vectors Computed in Frame 1 Instead of Frame O

A natural question to ask is how the translational part of the pose change parameters are affected by changing the observer from O to 1. One way to calculate this would be to first convert ${}^{O}_{A}\mathbf{t}_B$ to ${}^{O}_{A}\mathbf{d}_B$ and use similarity transformation. Then (25) could be calculated and then the resulting ${}^{1}_{A}\mathbf{d}_B$ could be converted ${}^{1}_{A}\mathbf{t}_B$.

Rather than doing this, we can calculate a clean expression for ${}^{1}_{A}\mathbf{t}_B$ directly, cf. Figure 6. Since

$${}^{1}_{A}\mathbf{t}_B = {}^{1}_{1}R_A\, {}^{A}_{A}\mathbf{t}_B \text{ and } {}^{O}_{A}\mathbf{t}_B = {}^{O}_{O}R_A\, {}^{A}_{A}\mathbf{t}_B$$

we have

$${}^{1}_{1}R_A^T\, {}^{1}_{A}\mathbf{t}_B = {}^{A}_{A}\mathbf{t}_B = {}^{O}_{O}R_A^T\, {}^{O}_{A}\mathbf{t}_B.$$

Therefore,

$${}^{1}_{A}\mathbf{t}_B = {}^{1}_{1}R_A\, {}^{O}_{O}R_A^T\, {}^{O}_{A}\mathbf{t}_B.$$

This can be written as

$${}^{1}_{A}\mathbf{t}_B = {}^{1}_{1}R_O\, {}^{O}_{A}\mathbf{t}_B. \tag{27}$$

Combining (26) and (27) and using (24) gives

$$\boxed{\left({}^{1}_{A}R_B, {}^{1}_{A}\mathbf{t}_B\right) = \left({}^{1}_{1}R_0, \mathbf{0}\right) \odot \left({}^{O}_{A}R_B, {}^{O}_{A}\mathbf{t}_B\right).} \tag{28}$$

6.2 CHANGE OF BODY-FIXED FRAME

In this subsection, we think of pose change as modeling the motion of a frame that is attached to a rigid body. This means

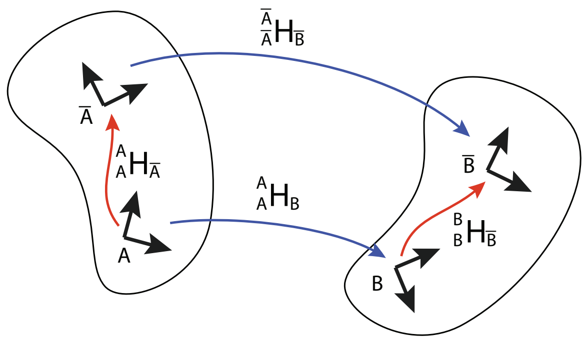

FIGURE 7. Conjugation Resulting from Changing Body-Fixed Frames

that we interpret frame A as where that body-fixed frame is before motion of the body and frame B as where it goes to after motion. We now ask the question what happens when we change the body-fixed frame representing the rigid body from A to $\overline{A}$ before motion resp. from B to $\overline{B}$ after motion. Since the relative pose of $\overline{A}$ to A as seen in A is the same as the pose of $\overline{B}$ relative to B as seen in B, we can say

$$ {}^A_AH_{\overline{A}} = {}^B_BH_{\overline{B}} \doteq \Delta $$

and

$$ {}^{\overline{A}}_{\overline{A}}H_{\overline{B}} = \Delta^{-1}\,{}^A_AH_B\,\Delta \tag{29} $$

as is clear from Figure 7. Then,

$$ {}^O_{\overline{A}}H_{\overline{B}} = {}^O_OH_{\overline{A}}\,{}^{\overline{A}}_{\overline{A}}H_{\overline{B}}\,{}^O_OH_{\overline{A}}^{-1}. $$

Let R_0 be implicitly defined by

$$ {}^O_OH_{\overline{A}}\Delta^{-1} = H(R_0,\mathbf{d}_0). $$

Then, using the direct-product calculus, the rotational parts follow in the same way as above, and Figure 8 shows how the translational part of the pose changes behave. Putting these parts together gives the direct-product rule

$$ \boxed{\left({}^O_{\overline{A}}R_{\overline{B}},{}^O_{\overline{A}}\mathbf{t}_{\overline{B}}\right) = \left(R_0,{}^O_B\mathbf{t}_{\overline{B}}\right)\bullet\left({}^O_AR_B,{}^O_A\mathbf{t}_B\right)\bullet\left(R_0,{}^O_A\mathbf{t}_{\overline{A}}\right)^{-1},} \tag{30} $$

which is not a conjugation as in SE(3), but is equally convenient.

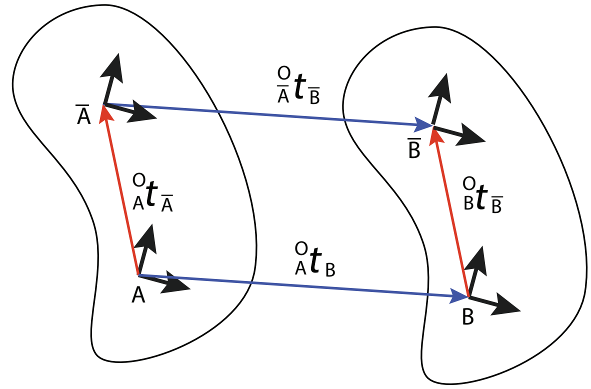

FIGURE 8. ${}^O_A\mathbf{t}_{\overline{A}} + {}^O_{\overline{A}}\mathbf{t}_{\overline{B}} = {}^O_A\mathbf{t}_B + {}^O_B\mathbf{t}_{\overline{B}}$.

Remark 6.1. *Two distinct actions of* PCG(3) *on Euclidean space can be defined:*

$$ \begin{aligned} (R,\mathbf{t}) \odot_1 \mathbf{x} &\doteq \mathbf{t}+\mathbf{x}, \\ (R,\mathbf{t}) \odot_2 \mathbf{x} &\doteq R\mathbf{x}. \end{aligned} \tag{31} $$

Both $\odot_1$ *and* $\odot_2$ *satisfy the definition of an action in (3). Note that* $\left({}^O_AR_B,{}^O_A\mathbf{t}_B\right)\odot_2\,{}^O_A\mathbf{t}_X = {}^O_B\mathbf{t}_{X'}$. *It then follows that*

$$ \left({}^O_AR_B,{}^O_A\mathbf{t}_B\right)\odot_1\left(\left({}^O_AR_B,{}^O_A\mathbf{t}_B\right)\odot_2\,{}^O_A\mathbf{t}_X\right) = {}^O_A\mathbf{t}_{X'}\,. $$

The effect of the above is the same as the SE(3) *action describing Euclidean motion in (2), although this construction is artificial in the context of pose changes modeled by* PCG(3) *acting on pose space.*

7 APPLICATIONS

As motivated in the introduction, an observer at frame O will observe the pose change from A to B in terms of the parameters $({}^O_AR_B,{}^O_A\mathbf{t}_B)$ rather than the Euclidean displacement parameters ${}^O_AH_B = H({}^O_AR_B,{}^O_A\mathbf{d}_B)$. This is the main reason for our exploration of this approach. A consequence is the direct-product structure and associated actions and transformation rules discussed in the previous section. Below we show that this endows $PCG(3)$ with a bi-invariant metric.

7.1 BI-INVARIANT METRICS

It has been well-known for quite some time that $SE(3)$ does not have nontrivial bi-invariant distance/metric functions, [16, 18, 23, 26]. Yet measuring distance between poses remains important both in kinematics [20,21] and in motion planning [2]. Therefore, either left-invariant or right-invariant metrics are chosen. Usually, left-invariance is more meaningful. When making this choice, the lack of bi-invariance means that

$$d_{SE(3)}(H_0H_1, H_0H_2) = d_{SE(3)}(H_1, H_2)$$

but

$$d_{SE(3)}(H_0H_1H_0^{-1}, H_0H_2H_0^{-1}) \neq d_{SE(3)}(H_1, H_2) \neq d_{SE(3)}(H_1H_0, H_2H_0)$$

for general homogeneous transformations $H_0, H_1, H_2 \in SE(3)$. Though $SE(3)$ metrics can exhibit bi-invariance for special pairs of motions [10], it would be nice if bi-invariance could be used as a general feature of metrics. Unfortunately this is not possible.

However, the direct product structure does tolerate bi-invariance. Namely, if

$$d_{\mathbb{R}^3}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_p$$

is any p-norm, it is invariant under translational shifts. If $d_{SO(3)}(R_1, R_2)$ is any bi-invariant norm for $SO(3)$ (such as the Frobenius norm $\|R_1 - R_2\|_F$ or $\|\log(R_1^T R_2)\|$, then

$$d_{PCG(3)}((R_1, \mathbf{t}_1), (R_2, \mathbf{t}_2)) = d_{\mathbb{R}^3}(\mathbf{x}, \mathbf{y}) + d_{SO(3)}(R_1, R_2)$$

will be bi-invariant under the direct product. That is

$$d_{PCG(3)}((R_0, \mathbf{t}_0) \bullet (R_1, \mathbf{t}_1), (R_0, \mathbf{t}_0) \bullet (R_2, \mathbf{t}_2)) = d_{PCG(3)}((R_1, \mathbf{t}_1), (R_2, \mathbf{t}_2)) = d_{PCG(3)}((R_1, \mathbf{t}_1) \bullet (R_0, \mathbf{t}_0), (R_2, \mathbf{t}_2)) \bullet (R_0, \mathbf{t}_0)).$$

Metrics have a role in interpolating paths between poses (or motions). The next subsection illustrates how motions observed from O are interpolated.

7.2 PATH GENERATION

Consider the task of an observer designing a trajectory to take frame A to frame B where the design is undertaken in the observer's frame of reference O. Applying the SE(3) machinery and using geodesic interpolation one obtains a trajectory that starts at ${}^O_O H_A$ at $\tau = 0$ and ends at ${}^O_O H_B$ at $\tau = 1$ as

$$H(\tau) = \exp\left(\tau \log({}^O_A H_B)\right) {}^O_O H_A .$$

Here the $SE(3)$ exponential and logarithm are matrix operations. This generates the helical tracjectory shown in Figure 9. In contrast, for $PCG(3)$, the analogous approach leads to

$$(R(\tau), \mathbf{t}(\tau)) = \left(\exp\left(\tau \log({}^O_A R_B)\right), {}^O_A \mathbf{t}_B \tau\right) \bullet \left({}^O_O R_A, {}^O_O \mathbf{t}_A\right),$$

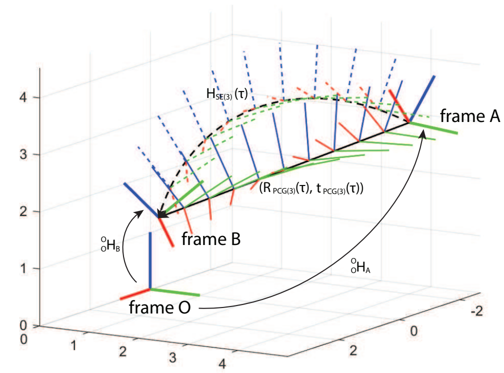

FIGURE 9. Comparison of Geodesic Trajectories for $SE(3)$ and $PCG(3)$

TABLE 1. Start and end poses of the moving frame (relative to O)

Poses	Rotations		Translations
	Axes of Rotations	Angles (rad)	
A	[0.1102; 0.9939; 0.0048]	$\frac{\pi}{4}$	[-2; 4; 3]
B	[0.0750; 0.6763; 0.0033]	$-\frac{\pi}{3}$	[3; 2; 2]

the straight line trajectory shown in Figure 9. The configurations of frame A and frame B used in the figure are summarized in Table 1.

Both the SE(3) or PCG(3) trajectories may be appropriate in different contexts, however, in many kinematics applications it would seem more natural to use trajectories generated by the pose change group PCG(3) carrying a bi-invariant metric.

8 CONCLUSIONS

Whereas in the kinematics of machines and linkages, changes in pose are indistinguishable from Euclidean motions, when viewed from an independent third frame pose changes and Euclidean motions obey different composition and transformation rules. This paper articulates these subtle differences. The calculus of Euclidean motions and how they change with perspective and under coordinate changes is well established. This is reviewed here to compare and contrast with our contribution, which is the development of an analytical framework for the composition of pose changes as a direct product operation, together with direct product actions and rules for changes of perspective changes. Implications for bi-invariant metrics are also explored.

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