Partial Bi-Invariance of SE(3) Metrics[1]

Gregory S. Chirikjian
Robot and Protein Kinematics Lab,
Department of Mechanical Engineering,
Johns Hopkins University,
Baltimore, MD 21218
e-mail: gregc@jhu.edu

In a flurry of articles in the mid to late 1990s, various metrics for the group of rigid-body motions, SE(3), were introduced for measuring distance between any two reference frames or rigid-body motions. During this time, it was shown that one can choose a smooth distance function that is invariant under either all left shifts or all right shifts, but not both. For example, if one defines the distance between two reference frames to be an appropriately weighted Frobenius norm of the difference of the corresponding homogeneous transformation matrices, this will be invariant under left shifts by arbitrary rigid-body motions. However, this is not the full picture—other invariance properties exist. Though the Frobenius norm is not invariant under right shifts by arbitrary rigid-body motions, for an appropriate weighting it is invariant under right shifts by pure rotations. This is also true for metrics based on the Lie-theoretic logarithm. This paper goes further to investigate the full invariance properties of distance functions on SE(3), clarifying the full subsets of motions under which both left and right invariance is possible.
[DOI: 10.1115/1.4028941]

Introduction

The concept of distance between rigid-body motions (or poses) arises in a variety of applications ranging from robotics, to computer graphics, and Computer-Aided Design/Computer-Aided Manufacturing (CAD/CAM). These application include partitioning rigid-body trajectories evenly, interpolating motions, and defining cost functions in rigid-body motion based on metrics to tackle problems in attitude estimation and sensor calibration.

Given homogeneous (rigid-body) transformations of the form[2]

$$H_i = H(R_i, \mathbf{t}_i) = \begin{pmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^{\mathrm{T}} & 1 \end{pmatrix} \in \mathrm{SE}(3)$$

a bona fide distance function, $d : \mathrm{SE}(3) \times \mathrm{SE}(3) \rightarrow \mathbb{R}_{\geq 0}$ is one that satisfies the properties

$$d(H_1, H_2) = 0 \Leftrightarrow H_1 = H_2 \quad (1)$$

$$d(H_1, H_2) = d(H_2, H_1) \quad (2)$$

$$d(H_1, H_2) + d(H_2, H_3) \geq d(H_1, H_3) \quad (3)$$

This is the general definition of a metric function, which can be defined for any space, not only SE(3). Since SE(3) is known to be a Lie group, it is desirable to consider metrics that possess additional invariance properties under simultaneous shifts of their arguments.

A number of papers over the past twenty years have introduced various metrics to define appropriate invariant concepts of distances. The study of distance functions for SE(3) originated in the mechanisms community and includes approaches based on Frobenius norms of differences of homogeneous transformations [1–4], Lie-theoretic logarithms of relative motions [5], stereographic projection into the four-sphere [6] akin to (and derived independently from) Ref. [7], metrics based on kinematic mappings [8], metrics based on convolution [4], and infinitesimal differential-geometric approaches [9,10]. Some time after this, computer scientists became interested in the problem of defining (or redefining) such metrics for robot motion planning [11,12]. Interest in the topic has persisted and other new approaches, such as using the polar and singular value decompositions have been presented rather recently [13]. For more comprehensive reviews, see Refs. [14] and [15], and for a review of metrics for the pure rotation case (where full bi-invariance is possible), see Ref. [16].

For example, many metrics that will be reviewed below are left invariant, meaning that[3]

$$d_{\mathrm{l}}(H_3 H_1, H_3 H_2) = d_{\mathrm{l}}(H_1, H_2) \quad (4)$$

for every $H_1, H_2, H_3 \in \mathrm{SE}(3)$. Alternatively, it is possible to define right-invariant metrics that satisfy the condition

$$d_{\mathrm{r}}(H_1 H_3, H_2 H_3) = d_{\mathrm{r}}(H_1, H_2) \quad (5)$$

A simple trick can be used to create a right- or left-invariant metric from a metric without any invariance as follows. Let $d_{\mathrm{r}}(H_1, H_2) \doteq d(\mathbb{I}_4, H_2 H_1^{-1})$ and $d_{\mathrm{l}}(H_1, H_2) \doteq d(\mathbb{I}_4, H_1^{-1} H_2)$, where $d(\cdot,\cdot)$ is a metric without invariance properties, and $\mathbb{I}_4$ is the 4×4 identity matrix that serves as the group identity for SE(3).

Curiously, in general it is not possible to satisfy both invariances simultaneously for arbitary $H_1, H_2, H_3 \in \mathrm{SE}(3)$. However, for pure rotations, the situation is quite different. In this case, it is possible to define measures of distance that are invariant under both left and right shifts. That is, for arbitrary $R_1, R_2, R_3 \in \mathrm{SO}(3)$, it is possible to define bona fide continuous "bi-invariant" distance functions $d_{\mathrm{b}} : SO(3) \times SO(3) \rightarrow \mathbb{R}_{\geq 0}$ that do satisfy

$$d_{\mathrm{b}}(R_1 R_3, R_2 R_3) = d_{\mathrm{b}}(R_3 R_1, R_3 R_2) = d_{\mathrm{b}}(R_1, R_2) \quad (6)$$

As a side note, it is worth mentioning that the word "continuous" is critical here because without it, a discontinuous trivial metric on any group, G, can be defined as

$$d_{\mathrm{trivial}}(g_1, g_2) \doteq \begin{cases} 0 & \text{if} \quad g_1 = g_2 \\ 1 & \text{if} \quad g_1 \neq g_2 \end{cases}$$

and this is bi-invariant. But when G is not compact and not commutative, and is not the direct product of compact and

[1]This paper was originally presented at the ASME 2014 International Design Engineering Technical Conferences as Paper No. DETC2014-DETC2014-34276.

[2]Here, R_i is a 3×3 rotation matrix and $\mathbf{t}_i$ is a three-dimensional translation vector.

Contributed by the Design Engineering Division of ASME for publication in the Journal of Computing and Information Science in Engineering. Manuscript received August 23, 2014; final manuscript received October 26, 2014; published online January 13, 2015. Editor: Bahram Ravani.

[3]Here and throughout this paper, juxtapositioning two homogeneous transformations implies matrix multiplication. That is, $H(R_1, \mathbf{t}_1) H(R_2, \mathbf{t}_2) = H(R_1 R_2, R_1 \mathbf{t}_2 + \mathbf{t}_1)$.

commutative groups, continuous bi-invariant metrics do not exist in general. And since SE(3) is a semidirect (rather than direct) product, there is no reason to believe that bi-invariant metrics should exist.

However, this is not the end of the story. Though it has been shown that it is not possible to define continuous distance functions for SE(3) that are bi-invariant, it is nevertheless possible to observe the following two facts:

(1) There exist left-invariant metrics for SE(3) for which

$$d_l(H_1K, H_2K) = d_l(H_1, H_2)$$

for any H_1, $H_2 \in$ SE(3) and $K = H(R_3, \mathbf{0})$ for any $R_3 \in$ SO(3); and

(2) Given any $H_3 \in$ SE(3), there exist special combinations of (not-necessarily commuting) motions H_1 and H_2 (which depend on H_3) for which the equality

$$d_l(H_1H_3, H_2H_3) = d_l(H_1, H_2)$$

holds.

The combination of these two observations form the core of this paper, which is organized as follows. First, a detailed review of various metrics from the literature is presented. Then the partial bi-invariance properties of SE(3) metrics are derived. And finally, some ramifications of these invariances are presented.

A Survey of Metrics

This section begins with metrics for SO(3) and then moves to the SE(3) case.

The SO(3) Case. The group SO(3) is a compact connected three-dimensional matrix Lie group defined by the conditions

$$\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times3} | RR^{\mathrm{T}} = \mathbb{I}_3, \det R = +1\}$$

Metrics for SO(3) can be defined from any matrix norm as

$$d_N(R_1, R_2) \doteq \| R_1 - R_2 \|_N$$

where N indicates that it is defined relative to a particular norm, $\| \cdot \|_N$. For example, if the Frobenius norm is used, then

$$d_F(R_1, R_2) = \| R_1 - R_2 \|_F \doteq \sqrt{\mathrm{tr}((R_1 - R_2)(R_1 - R_2)^{\mathrm{T}})}$$

and from the fact that the trace is invariant under similarity transformation, it is easy to see that

$$d_F(R_3R_1, R_3R_2) = \sqrt{\mathrm{tr}(R_3(R_1 - R_2)(R_1 - R_2)^{\mathrm{T}}R_3^{\mathrm{T}})} = d_F(R_1, R_2)$$

And on the other side

$$d_F(R_1R_3, R_2R_3) = \sqrt{\mathrm{tr}((R_1 - R_2)R_3R_3^{\mathrm{T}}(R_1 - R_2)^{\mathrm{T}})} = d_F(R_1, R_2)$$

This demonstrates the existence of bi-invariant metrics for SO(3), which is a property that would also hold if the induced-2-norm $\| \cdot \|_2$ had been used in place of $\| \cdot \|_F$. But this does not imply that every distance function on SO(3) is bi-invariant (or even invariant to shifts on one side).

For example, if instead of the usual Frobenius norm, a weighted Frobenius norm of the form

$$d_W(R_1, R_2) = \| R_1 - R_2 \|_W \doteq \sqrt{\mathrm{tr}((R_1 - R_2)W(R_1 - R_2)^{\mathrm{T}})}$$

is used, where $W = W^{\mathrm{T}}$ is positive definite but without additional structure, then the result will still be a valid distance function. But the right invariance that existed with the identity-weighted Frobenius norm will no longer hold for arbitrary right shifts (though it may for special values of R_3 if W has special structure). Right-invariant metrics for SO(3) that lack left invariance can be constructed just as easily. And an example of a metric that is neither left- nor right invariant, consider $d_1(R_1, R_2) = \| R_1 - R_2 \|_1$.

However, since SO(3) is a compact Lie group, it has a unique bi-invariant integration measure which can be used to average metrics that are not invariant, to produce ones that are invariant. Given a continuous function $f : \mathrm{SO}(3) \to \mathbb{R}$, there is essentially only one "correct" (bi-invariant) way to compute the integral of that function. Explicitly, if R is expressed in ZXZ Euler angles α, β, γ, this is

$$\int_{\mathrm{SO}(3)} f(R)dR = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(R(\alpha, \beta, \gamma)) \sin\beta d\alpha d\beta d\gamma$$

(Sometimes a normalizing factor of $1/8\pi^2$ is introduced so that $d'R = (1/8\pi^2) \sin\beta d\alpha d\beta d\gamma$ and $\int_{\mathrm{SO}(3)} 1 d'R = 1$, but this is only a matter of convention and not an independent choice of integration measure.)

The bi-invariance of this integration measure means that

$$\int_{\mathrm{SO}(3)} f(R)dR = \int_{\mathrm{SO}(3)} f(QR)dR = \int_{\mathrm{SO}(3)} f(RQ)dR$$

for arbitrary fixed $Q \in$ SO(3). And so, given a continuous left-invariant metric, it is always possible to define

$$d_b(R_1, R_2) = \int_{\mathrm{SO}(3)} d_l(R_1Q', R_2Q')dQ' \tag{7}$$

This averaging on the right together with existing invariance of integration under left shifts gives a new metric that is invariant under both left and right shifts. (Averaging a metric without any invariance properties by shifting on the left and on the right, and integrating over two copies of SO(3) can also be done to create a bi-invariant metric.) The construct in Eq. (7) works because all values of d_l are finite (since SO(3) is compact and d_l is taken to be continuous) and the volume of SO(3) is finite (again from its compactness). But this argument will not work for integrating over SE(3), which is not compact.

From Euler's theorem, it is known that any rotation matrix can be written as

$$R(\theta, \mathbf{n}) = \mathbb{I}_3 + \sin\theta N + (1 - \cos\theta)N^2 = \exp(\theta N) \tag{8}$$

where θ is the angle of rotation, and $N = \hat{\mathbf{n}}$ is the unique skew-symmetric matrix corresponding to the unit vector defining the axis of rotation, $\mathbf{n} = [n_1, n_2, n_3]^{\mathrm{T}}$ such that $N\mathbf{x} = \mathbf{n} \times \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. The angle can be obtained from a given R by either taking the log and then computing the Frobenius norm of the result, or by directly manipulating the above axis-angle formula to isolate θ when it is in the range (0, π). The result is the same.

Given two rotations, the angle of the relative rotation between them is a measure of distance. That is,

$$d_{\log}(R_1, R_2) \doteq \theta(R_1^{\mathrm{T}}R_2) = \frac{1}{2} \| \log(R_1^{\mathrm{T}}R_2) \|_F \tag{9}$$

is a valid distance function [5] which is naturally left invariant. Moreover, since $\log(QRQ^{\mathrm{T}}) = Q(\log R)Q^{\mathrm{T}}$, this metric is bi-invariant.

Metrics defined using norms applied directly to the difference of two rotation matrices are sometimes referred to as being "extrinsic" and those using the log of the relative rotation are called "intrinsic."

The SE(3) Case. Several complications present themselves in the case of full rigid-body motions consisting of both rotations and translations. The first is that there are no continuous metric functions that are invariant under arbitrary left and right shifts. The second is that since rotations are measured in one kind of units (radians, degrees, etc.), and translations are measured in units of length (inches, meters, etc.), combining both into one number must be done with caution.

Whereas the former is something that cannot be changed, the latter was addressed in a satisfactory way in several papers [1–4]. The argument goes as follows. Since there is no problem measuring distance between points in Euclidean space by simply computing the Euclidean norm of their difference, $d(\mathbf{x}_1,\mathbf{x}_2) = \| \mathbf{x}_1 - \mathbf{x}_2 \|$, and since by definition this is left invariant under Euclidean motions

$$d(H\cdot\mathbf{x}_1, H\cdot\mathbf{x}_2) = d(\mathbf{x}_1,\mathbf{x}_2)$$

where $H\cdot\mathbf{x}_1 = R\mathbf{x}_1 + \mathbf{t}$, then

$$d_{\mathrm{l}}(H_1,H_2) \doteq \left(\int_{\mathbb{R}^n} \| H_1\cdot\mathbf{x} - H_2\cdot\mathbf{x} \|^2 \rho(\mathbf{x})d\mathbf{x}\right)^{1/2} \tag{10}$$

will inherit this left invariance. Here, $\rho(\mathbf{x})$ is a density function that decays rapidly to zero away from the origin. For example, this could be the mass density of a finite body that takes a value of 1 on the body and zero outside of it.

The reason for squaring the norm inside the integral in Eq. (10) is that the result can be integrated in closed form. But this by itself kills the triangle inequality. Taking the square-root outside recovers the triangle inequality [4].

It has been shown that when $\int_{\mathbb{R}^n}\mathbf{x}\rho(\mathbf{x})d\mathbf{x} = \mathbf{0}$, Eq. (10) is completely equivalent to the weighted Frobenius norm [1–4]

$$d_{\mathrm{l}}(H_1,H_2) = \| H_1 - H_2 \|_W$$

where

$$W = \begin{pmatrix} J & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & m \end{pmatrix} \tag{11}$$

where $m = \int_{\mathbb{R}^n}\rho(\mathbf{x})d\mathbf{x}$ and $J = \int_{\mathbb{R}^n}\mathbf{x}\mathbf{x}^{\mathrm{T}}\rho(\mathbf{x})d\mathbf{x}$ are, respectively, mass and inertia matrices. Note that J is related to the usual moment of inertia matrix, I, through the expression

$$I = \mathrm{tr}(J)\mathbb{I}_3 - J$$

The combination of m and J naturally weights translations and rotations according to the properties of the body being moved without the need to artificially introduce a length parameter. A body that is very concentrated will have small J, and one that is very spread out will have large J. For example, for a point mass, J will be zero, and for a long rod J will be large, while m will be the same in both cases. This reflects that for a long rod, the "cost" of rotation should be large, whereas rotation of a point mass around itself has no cost. In general, the metric in Eq. (10) is not right invariant.

Invariance of Existing SE(3) Metrics Under Left–Right SE(3) × SO(3) Actions

Note that when the body being moved is isotropic, so that for some $\alpha\in\mathbb{R}_{>0}$ that reflects how concentrated the density ρ is, and some characteristic length scale of the body, r, the inertia matrix will be of the form $J = \alpha m r^2\cdot\mathbb{I}_3$.

In the case of an isotropic body, the metric in Eq. (10) also has the property that when letting $K = H(R,\mathbf{0})$

$$d_{\mathrm{l}}(H_1K, H_2K) = d_{\mathrm{l}}(H_1,H_2)$$

for any $R\in\mathrm{SO}(3)$ and any H_1, $H_2\in\mathrm{SE}(3)$. Other metrics for SE(3) that share this property of full left invariance and partial right invariance include

$$d^{(1)}_{\mathrm{SO}(3),r}(H_1,H_2) \doteq r\cdot d_{\log}(R_1,R_2) + \| \mathbf{t}_1 - \mathbf{t}_2 \|$$

and

$$d^{(2)}_{\mathrm{SO}(3),r}(H_1,H_2) \doteq \sqrt{r^2\cdot d^2_{\log}(R_1,R_2) + \| \mathbf{t}_1 - \mathbf{t}_2 \|^2}$$

and

$$d_{\mathrm{SE}(3),r}(H_1,H_2) \doteq \| \log(H_1^{-1}H_2) \|_W$$

where W is as in Eq. (11) but for the special case of an isotropic body.

Another class of metrics is defined by shifting a unimodal probability density function of motion, $f: SE(3)\to\mathbb{R}_{\geq 0}$, and integrating. For example,

$$d_{\mathrm{f}}(H_1,H_2) \doteq \left(\int_{\mathrm{SE}(3)} |f(H_1^{-1}H) - f(H_2^{-1}H)|^2 dH\right)^{1/2}$$

The integration measure for SE(3) is bi-invariant, as is the SO(3) integration measure. And so letting $H' = H_3^{-1}H$

$$d_{\mathrm{f}}(H_3H_1,H_3H_2) \doteq \left(\int_{\mathrm{SE}(3)} |f(H_1^{-1}H') - f(H_2^{-1}H')|^2 dH'\right)^{1/2} = d_{\mathrm{f}}(H_1,H_2)$$

It is often a point of confusion in the kinematics literature that SE(3) can have a bi-invariant integration measure but not a bi-invariant metric. The bi-invariant integration measure for $H = H(R,\ \mathbf{t})$ is simply $dH = dRd\ \mathbf{t}$, where dR is the bi-invariant integration measure for SO(3) described earlier, and $dt = dt_1\ dt_2\ dt_3$. The proof that this is bi-invariant is given in Ref. [14]. Maybe the simplest way to say it is that conditions for the metric tensor $G(R,\ \mathbf{t})$ to be bi-invariant are more stringent than for the integration measure, which is more akin to $|\det G(R,\mathbf{t})|$. The former can be viewed as a 6×6 matrix function with 21 independent entries whereas the latter is a single scalar function which is invariant under shifts by unimodular matrices and arbitrary similarity transformations.

The metrics $d_f(H_1,\ H_2)$ do not appear to have the right invariance under SO(3). But given a left-invariant metric on SE(3) that does not satisfy this condition of partial right invariance, it is possible to average over rotations on the right as in Eq. (7) to form a new metric with right invariance under SO(3) actions. It is not possible to average over all of SE(3) because it is not compact, and the result will not be finite.

Analysis of Full Invariance

In this section, two questions are addressed in the context of SE(3) metrics that are simultaneously left invariant under SE(3) and right invariant under SO(3). Given the discussion above, there is no loss of generality in assuming this. Both questions query special cases when $d_{\mathrm{l}}(H_1\ H_3,\ H_2\ H_3) = d_{\mathrm{l}}(H_1,\ H_2)$ where $H_3\in\mathrm{SE}(3)$ is not a pure rotation. Namely,

(1) For a pair of arbitrary homogeneous transformations, (H_1, H_2), what is the largest subset of SE(3) from which right shifts H_3 can be drawn and applied such that the value of the metric is unchanged?
(2) For a given right shift by an arbitrary rigid-body motion, H_3, what is the largest set of pairs (H_1, H_2) for which the value of the metric is unchanged?

The answers to both of these questions are addressed below.

Answer to Question 1. Addressing question 1 first, for given (H_1, H_2), consider under what conditions ($H_1 H_3$, $H_2 H_3$) can be written in the form $(H_3' H_1 K, H_3' H_2 K)$ where $K = H(R, \mathbf{0})$. Since from the discussion in the previous section, SE(3) metrics can be assumed to have left–right $\mathrm{SE}(3)\times\mathrm{SO}(3)$ invariance, $d(H_3'H_1K, H_3'H_2K) = d(H_1, H_2)$. And from the symmetry of a metric, this is also equal to $d(H_2, H_1)$. Therefore, consider the two subquestions:

(1a) Under what conditions on H_3 is it possible to find H_3' and K for which the following equations will hold simultaneously for given (H_1, H_2)?

$$H_3'H_1K = H_1H_3 \quad \text{and} \quad H_3'H_2K = H_2H_3 \tag{12}$$

(1b) Under what conditions on H_3 is it possible to find H_3'' and K' for which the following equations will hold simultaneously for given (H_1, H_2)?

$$H_3''H_1K' = H_2H_3 \quad \text{and} \quad H_3''H_2K' = H_1H_3 \tag{13}$$

Computing the solution spaces for both problems will each provide branches of possible solutions, the union of which will describe the full space of H_3's for which right invariance will hold.

Rewriting Eq. (12) so as to isolate H_3 in both equations, the condition

$$H_1^{-1}H_3'H_1K = H_3 = H_2^{-1}H_3'H_2K$$

emerges, giving the condition $H_1^{-1}H_3'H_1 = H_2^{-1}H_3'H_2$. In other words, K plays no role in constraining H_3'. The result can further be written as

$$(H_2H_1^{-1})H_3' = H_3'(H_2H_1^{-1})$$

This is a single instance of what is known as the "$AX = XB$ sensor calibration problem" [17–20] where here $X = H_3'$, and in this case

$$A = B = H_2H_1^{-1}$$

Depending on the specifics of A and B, the solution space for X can either be empty or it can be two-, four-, or six-dimensional. When $A = B$, solutions always exist, and when $A = B = \mathbb{I}_4$ then obviously X can take any value in SE(3), which is six-dimensional. The precise conditions distinguishing the remaining two cases are most easily stated by writing A (and hence B) in terms of their screw-theoretic decomposition. Let

$$\mathrm{screw}(\mathbf{n}, \mathbf{p}, \theta, d) \doteq \begin{pmatrix} e^{\theta N} & (\mathbb{I}_3 - e^{\theta N})\mathbf{p} + d\mathbf{n} \\ \mathbf{0}^T & 1 \end{pmatrix} \tag{14}$$

where $N = -N^{\mathrm{T}}$ is the skew-symmetric matrix such that $N\mathbf{x} = \mathbf{n} \times \mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^3$ and $\mathbb{I}_3$ is the 3×3 identity matrix. $\mathbf{n}$ is the direction of the axis of rotation of $e^{\theta N}$ and $\mathbf{p}$ is the position of any point on the screw axis (i.e., not necessarily the usual Plücker coordinates). Then

$$\mathrm{screw}(\mathbf{n}, \mathbf{p}, \theta, d) = \mathrm{screw}(\mathbf{n}, \mathbf{p} + t\mathbf{n}, \theta, d)$$

for all $t \in \mathbb{R}$. Moreover, let

$$A = \mathrm{screw}(\mathbf{n}_A, \mathbf{p}_A, \theta_A, d_A)$$

and similarly for B. Then let $X = (R_X, \mathbf{t}_X) \in \mathrm{SE}(3)$. Multiplying homogenous transformations gives

$$X^{-1}\mathrm{screw}(\mathbf{n}, \mathbf{p}, \theta, d)X = \mathrm{screw}(R_X\mathbf{n}, R_X\mathbf{p} + \mathbf{t}_X, \theta, d) \tag{15}$$

where $X \cdot (\mathbf{n}, \mathbf{p}) \doteq (R_X\mathbf{n}, R_X\mathbf{p} + \mathbf{t}_X)$ defines the action of SE(3) on a line defined by ($\mathbf{n}$, $\mathbf{p}$). (If $\mathbf{p}$ had been chosen as in Plücker coordinates, then the action $X \cdot (\mathbf{n}, \mathbf{p})$ would necessarily be more complicated to ensure that the constraint $\mathbf{n} \cdot \mathbf{p} = 0$ is preserved after the action by X.) In the problem at hand, the line of interest is a screw axis of another rigid motion. Note that in Eq. (15) θ and d remain unchanged under conjugation. This is a statement of the well-known fact that these are SE(3) invariants [21,22]. This indicates that if A and B are related as $AX = XB$, then they must have the same invariants

$$\theta_A = \theta_B \quad \text{and} \quad d_A = d_B \tag{16}$$

This condition can be called the "compatibility of A and B."

The question then becomes, if Eq. (16) holds for given A and B, what is the space of all possible X's for which $A = XBX^{-1}$? The search for an appropriate X can begin by taking the SE(3) logarithm of this equation, which can be rewritten as [14,15]

$$\log^\vee(A) = Ad(X)\log^\vee(B) \tag{17}$$

where for a general homogeneous transformation, $H \in \mathrm{SE}(3)$, the adjoint matrix is

$$Ad(H(R, \mathbf{t})) = \begin{pmatrix} R & \mathbb{O} \\ \hat{t}R & R \end{pmatrix}$$

In the case of general compatible A and B, i.e., not degenerate cases in which the rotation angle[4] is outside of the range $(0, \pi)$, the solution space of all possible X's that satisfy this equation is known to be two-dimensional. This can be seen by defining

$$\log^\vee(A) = \begin{pmatrix} \theta_A\mathbf{n}_A \\ \mathbf{v}_A \end{pmatrix} \tag{18}$$

Then Eq. (17) can be broken up into rotational and translational parts as

$$\mathbf{n}_A = R_X\mathbf{n}_B \quad \text{and} \tag{19}$$

$$\mathbf{v}_A = \theta_B\hat{\mathbf{t}}_X R_X\mathbf{n}_B + R_X\mathbf{v}_B \tag{20}$$

The first of these equations has a one-dimensional solution space of the form $R_X = R(\mathbf{n}_B, \mathbf{n}_A)R(\mathbf{n}_B, \phi)$, where $\phi \in [0, 2\pi)$ is free and $R(\mathbf{n}_B, \mathbf{n}_A)$ is any rotation matrix that rotates the vector $\mathbf{n}_B$ into $\mathbf{n}_A$. In particular, choose

$$R(\mathbf{n}_B, \mathbf{n}_A) = \mathbb{I} + \widehat{\mathbf{n}_B \times \mathbf{n}_A} + \frac{(1 - \mathbf{n}_B \cdot \mathbf{n}_A)}{\| \mathbf{n}_B \times \mathbf{n}_A \|^2}(\widehat{\mathbf{n}_B \times \mathbf{n}_A})^2 \tag{21}$$

The rotation $R(\mathbf{n}_B, \phi)$ is given by Euler's formula

[4]This angle is computed from the Frobenius norm $\theta_A = \| (1/2)\log R_A \| = \| (1/2)\log R_B \| = \theta_B$.

$$R(\mathbf{n}_B, \phi) = \mathbb{I} + \sin\phi\ \widehat{\mathbf{n}_B} + (1-\cos\phi)(\widehat{\mathbf{n}_B})^2$$

Substituting $R_X = R(\mathbf{n}_B, \mathbf{n}_A)\, R(\mathbf{n}_B, \phi)$ into Eq. (20) and rearranging terms

$$\frac{R(\mathbf{n}_B, \mathbf{n}_A) R(\mathbf{n}_B, \phi)\mathbf{v}_B - \mathbf{v}_A}{\theta_B} = \widehat{\mathbf{n}_A}\mathbf{t}_X \tag{22}$$

The skew-symmetric matrix $\widehat{\mathbf{n}_A}$ is rank 2, so a free translational degree of freedom exists in $\mathbf{t}_X$ along the $\mathbf{n}_A$ direction. $\mathbf{t}_X$ can thus be described as

$$\mathbf{t}_X = \mathbf{t}(s) = s\mathbf{n}_A + a\mathbf{m}_A + b\mathbf{m}_A \times \mathbf{n}_A \tag{23}$$

where $s \in \mathbb{R}$ is a second free parameter, and $\mathbf{m}_A$ and $\mathbf{m}_A \times \mathbf{n}_A$ are defined to be orthogonal to $\mathbf{n}_A$ by construction. If $\mathbf{n}_A = [n_1, n_2, n_3]^{\mathrm{T}}$ and n_1, n_2 are not simultaneously zero, then it is possible to define[5]

$$\mathbf{m}_A \doteq \frac{1}{\sqrt{n_1^2 + n_2^2}} \begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix}$$

The coefficients a and b are then computed by substituting Eq. (23) into Eq. (22) and using the fact that $\{\mathbf{n}_A, \mathbf{m}_A, \mathbf{n}_A \times \mathbf{m}_A\}$ is an orthonormal basis for $\mathbb{R}^3$. Explicitly

$$a = -\left(\frac{R(\mathbf{n}_B, \mathbf{n}_A)R(\mathbf{n}_B, \phi)\mathbf{v}_B - \mathbf{v}_A}{\theta_B}\right) \cdot (\mathbf{m}_A \times \mathbf{n}_A) \quad \text{and}$$
$$b = \left(\frac{R(\mathbf{n}_B, \mathbf{n}_A)R(\mathbf{n}_B, \phi)\mathbf{v}_B - \mathbf{v}_A}{\theta_B}\right) \cdot \mathbf{m}_A$$

This means that the feasible solutions can be completely parameterized as

$$X(\phi, s) = H(R(\mathbf{n}_B, \mathbf{n}_A)R(\mathbf{n}_B, \phi), \mathbf{t}(s)) \tag{24}$$

where $(\phi, s) \in [0, 2\pi) \times \mathbb{R}$ and $H(R, \mathbf{t}) \in \mathrm{SE}(3)$.

Let us call $X(\phi, s)$ the 2D "cylinder" of X's that solve the problem for a single pair (A, B) with $\theta_A = \theta_B \in (0, \pi)$. In contrast, when $\theta_A = \theta_B \in \{0, \pi\}$ the concept of logarithm used in Eq. (18) breaks down, and other methods must be used. It suffices to say that, in this case, the space of allowable X's grows to be four-dimensional.

From this, it is clear that in the context of the present discussion there is always a solution, and for all but very special choices of H_1 and H_2, there will be two degrees of freedom in H_3'.

Then, the full set of values of H_3 for which a left-invariant SE(3) metric will be right invariant can be parameterized as

$$H_3 = H_1^{-1} H_3' H_1 K$$

Since K played no role in computing the freedom in H_3', the dimensions add. In other words, K has the three dimensions of freedom of SO(3), and if H_3' has two additional independent degrees of freedom, then there is a whole five-dimensional subspace of SE(3) from which H_3 can be chosen for which the left-invariant metric will also be right invariant. In the degenerate case, when $\theta_A = \theta_B \in \{0, \pi\}$ then the space of possible H_3's for which right invariance holds becomes six-dimensional, meaning that for these special values of H_1 and H_2, the metric is fully bi-invariant.

[5]The special case when they are simultaneously zero is a set of measure zero, and hence is a rare event. Nevertheless, it is easy to handle, since in this case R_A is necessarily a rotation around $\mathbf{e}_3$.

Considering question 1b, and isolating H_3 in Eq. (13), the conditions for the existence of a right shift that will not change the value of the metric are

$$H_2^{-1} H_3'' H_1 K' = H_3 = H_1^{-1} H_3'' H_2 K'$$

Like in case 1a, K' can be canceled to give $H_2^{-1} H_3'' H_1 = H_1^{-1} H_3'' H_2$ and the resulting constraint equation on H_3'' is

$$(H_1 H_2^{-1}) H_3'' = H_3'' (H_2 H_1^{-1})$$

Here

$$A = H_1 H_2^{-1} \quad \text{and} \quad B = A^{-1}$$

and since the screw parameters θ and d for a motion and inverse motion are the same, which can be observed from

$$[\mathrm{screw}(\mathbf{n}, \mathbf{p}, \theta, d)]^{-1} = \mathrm{screw}(-\mathbf{n}, \mathbf{p}, \theta, d)$$

the necessary conditions for existence of solutions are satisfied. And a different cylinder of solutions for H_3' is obtained. From here, the second branch of the full 5D solution space for H_3 is obtained as

$$H_3 = H_2^{-1} H_3'' H_1 K'$$

The final question that can be asked in regard to question 1 is whether these two branches of solutions are distinct, or if they have some intersection. And if they intersect, what is that space of intersection?

The two solutions will intersect if for fixed H_1, H_2, H_3', K, it is possible to find H_3'' in the 2D subspace of SE(3) and $K' = H(R, \mathbf{0})$ with $R \in SO(3)$ such that

$$H_1^{-1} H_3' H_1 K = H_2^{-1} H_3'' H_1 K' \tag{25}$$

has a solution. And if every such instance of this constraint has a solution, then branch 2 contains branch 1. And if the roles of (H_3', K) and (H_3'', K') are reversed and it is shown that branch 1 also contains branch 2, then this would imply that both branches are the same.

Obviously, from Eq. (13) it is possible to rewrite Eq. (25) as

$$H_1^{-1} H_3' H_1 K = H_1^{-1} H_3'' H_2 K'$$

Canceling H_1^{-1} on the left and choosing $K = K'$ on the right

$$H_3' H_1 = H_3'' H_2$$

Therefore, given H_3'' it is always possible to find H_3', and vice versa. And so cases 1a and 1b produce exactly the same solution spaces, and do not represent different branches.

Answer to Question 2. The statement of question 2 uses many of the same equations, but with different variables held fixed. The question is stated as: For given arbitrary right shift H_3 what is the space of all H_1, H_2, for which the left-invariant metric remains unchanged? In analogy with question 1, this can be broken up into two subquestions:

(2a) For given H_3, what are all possible H_1 and H_2 such that H_3' and K can be found for which the following equations will hold simultaneously?

$$H_3' H_1 K = H_1 H_3 \quad \text{and} \quad H_3' H_2 K = H_2 H_3$$

(2b) For given H_3, what are all possible H_1, H_2, such that H_3'' and K' can be found for which the following equations will hold simultaneously?

$$H_3''H_1K' = H_2H_3 \quad \text{and} \quad H_3''H_2K' = H_1H_3 \qquad (26)$$

Starting with 2a, the equations can be written as

$$H_3'H_1 = H_1(H_3K^{-1}) \quad \text{and} \quad H_3'H_2 = H_2(H_3K^{-1})$$

Each of these is an example of $AX = XB$. In the first case $X = H_1$, and in the second $X = H_2$. The necessary conditions for solutions to exist are

$$\theta(H_3') = \theta(H_3K^{-1}) \quad \text{and} \quad d(H_3') = d(H_3K^{-1})$$

And for each pair (H_3', K) when $\theta \in (0, \pi)$, a two-dimensional cylinder of solutions will exist, and when $\theta \in \{0, \pi\}$ the dimension of the solution space increases as discussed earlier. Since H_3' has six degrees of freedom, K has three degrees of freedom, and two constraints are imposed, it follows that starting with any initial pair (H_1', H_2') for which $d_l(H_1'H_3, H_2'H_3) = d_l(H_1', H_2')$, a whole $6 + 3 - 2 = 7$ dimensional subspace of other pairs in $SE(3) \times SE(3)$ will also have this property. Moreover, if $H_3 = \text{screw}(\mathbf{n}, \mathbf{p}, \theta, d)$, then $H_1' = \text{screw}(\mathbf{n}, \mathbf{p}, \theta_1', d_1')$ and $H_2' = \text{screw}(\mathbf{n}, \mathbf{p}, \theta_2', d_2')$ will commute with H_3. The four free parameters $\theta_1', d_1', \theta_2', d_2'$ used to define initial starting points for (H_1', H_2') together with the seven degrees of freedom that can be constructed for each of these pairs indicates an 11-dimensional space inside of the 12-dimensional space $SE(3) \times SE(3)$ that is invariant under right shifts by general $H_3 \in SE(3)$. Of course, for special values of H_3, such as $H_3 = H(R, \mathbf{0})$, full 12-dimensional invariance is possible. But the problem addressed here is the general case.

As for question 2b, if in Eq. (26) H_2 is isolated from each equation and the results are set equal to each other, another kind of $AX = XB$ equation with $X = H_1$ results

$$[(H_3'')^2]H_1 = H_1[H_3(K')^{-1}H_3(K')^{-1}]$$

But in general $AX = XB \Leftrightarrow A^2X = XB^2$ and so

$$H_3''H_1 = H_1[H_3(K')^{-1}] \qquad (27)$$

Similarly, it is possible to isolate H_1 from both equations in Eq. (26) and write

$$[(H_3'')^{-2}]H_2 = H_2[K'H_3^{-1}K'H_3^{-1}] \Rightarrow (H_3'')^{-1}H_2 = H_2[K'H_3^{-1}]$$

And since in general $AX = XB \Leftrightarrow A^{-1}X = XB^{-1}$, this can be written as

$$H_3''H_2 = H_2[H_3(K')^{-1}]$$

which means that the conditions on H_2 are the same as those on H_1 in Eq. (27).

The conditions for existence of solutions for this case are expressed as

$$\theta(H_3'') = \theta(H_3(K')^{-1}) \quad \text{and} \quad d(H_3'') = d(H_3(K')^{-1})$$

and the same argument as in case 2a indicates that this is an 11-dimensional space.

Conclusions

It has been known for twenty years that metrics on the group of rigid-body motions can be chosen to be left invariant under arbitrary rigid-body motions. However, the fact that additional invariance exists for special motions applied on the right appears not to have been studied previously. This paper fills this gap in the literature by characterizing the full set of invariances of SE(3) metrics. It is shown that a five-dimensional subspace of right shifts can be applied to an arbitrary pair of motions that will leave left-invariant metrics unchanged. In addition, the space of special pairs of motions for which arbitrary left and right shifts can be applied for which distance is preserved is analyzed and is found to be 11-dimensional in the general case.

Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant No. IIS-1162095. The proofreading done by Mr. Joshua Davis, Mr. Qianli Ma, and Mr. Hau Sze is greatly appreciated.

Nomenclature

$\| A \|_F =$ the Frobenius norm of a matrix $A \in \mathbb{R}^{n\times n}, \| A \|_F^2 \doteq \text{tr}(AA^T)$
$\| A \|_N =$ an arbitrary matrix norm
$\| A \|_W =$ the weighted Frobenius norm with $W = W^T \in \mathbb{R}^{n\times n}$ positive definite, $\| A \|_W^2 \doteq \text{tr}(AWA^T)$
$\| A \|_2 =$ the induced 2-norm of a matrix
$d(\cdot,\cdot) =$ a metric/distance function
det = the determinant of a square matrix
$\mathbb{I}_n =$ the $n \times n$ identity matrix
$\mathbb{R}^n =$ n-dimensional Euclidean space
$\mathbb{R}^{n\times n} =$ the space of $n \times n$ matrices with real entries
$SE(n) =$ the special Euclidean group of n-dimensional space
$SO(n) =$ the group of rotations in n-dimensional Euclidean space
$\text{tr}(A) =$ the trace of a square matrix

References

[1] Kazerounian, K., and Rastegar, J., 1992, "Object Norms: A Class of Coordinate and Metric Independent Norms for Displacement," *Flexible Mechanisms, Dynamics and Analysis*, ASME DE 47, pp. 271–275.
[2] Martinez, J. M. R., and Duffy, J., 1995, "On the Metrics of Rigid Body Displacement for Infinite and Finite Bodies," ASME J. Mech. Des., **117**(1), pp. 41–47.
[3] Fanghella, P., and Galletti, C., 1995, "Metric Relations and Displacement Groups in Mechanism and Robot Kinematic," ASME J. Mech. Des., **117**(3), pp. 470–478.
[4] Chirikjian, G. S., and Zhou, S., 1998, "Metrics on Motion and Deformation of Solid Models," ASME J. Mech. Des., **120**(2), pp. 252–261.
[5] Park, F. C., 1995, "Distance Metrics on the Rigid-Body Motions With Applications to Mechanism Design," ASME J. Mech. Des., **117**(1), pp. 48–54.
[6] Etzel, K. R., and McCarthy, J. M., 1996, "Spatial Motion Interpolation in an Image Space of SO(4)," Proceedings of 1996 ASME Design Engineering Technical Conference and Computers in Engineering Conference, Irvine, CA, Aug. 18–22.
[7] Inonu, E., and Wigner, E. P., 1953, "On the Contraction of Groups and Their Representations," Proc. Natl. Acad. Sci., **39**(6), pp. 510–524.
[8] Ravani, B., and Roth, B., 1984, "Mappings of Spatial Kinematics," ASME J. Mech. Des., **106**(3), pp. 341–347.
[9] Žefran, M., Kumar, V., and Croke, C., 1999, "Metrics and Connections for Rigid-Body Kinematics," Int. J. Rob. Res., **18**(2), pp. 243–258.
[10] Lin, Q., and Burdick, J. W., 2000, "Objective and Frame-Invariant Kinematic Metric Functions for Rigid Bodies," Int. J. Rob. Res., **19**(6), pp. 612–625.
[11] Kuffner, J. J., 2004, "Effective Sampling and Distance Metrics for 3D Rigid Body Path Planning," Proceedings of the 2004 IEEE International Conference on Robotics and Automation, ICRA'04, New Orleans, LA, April, Vol. 4, pp. 3993–3998.
[12] Amato, N. M., Bayazit, O. B., Dale, L. K., Jones, C., and Vallejo, D., 1998, "Choosing Good Distance Metrics and Local Planners for Probabilistic Roadmap Methods," Proceedings of the 1998 IEEE International Conference on Robotics and Automation, ICRA'98, Leuven, Belgium, May, Vol. 1, pp. 630–637.
[13] Larochelle, P. M., Murray, A. P., and Angeles, J., 2007, "A Distance Metric for Finite Sets of Rigid-Body Displacements via the Polar Decomposition," ASME J. Mech. Des., **129**(8), pp. 883–886.
[14] Chirikjian, G. S., and Kyatkin, A. B., 2001, *Engineering Applications of Noncommutative Harmonic Analysis*, CRC Press, Boca Raton, FL.
[15] Chirikjian, G. S., 2009/2012, *Stochastic Models, Information Theory, and Lie Groups: Volumes I + II*, Birkhäuser, Boston, MA.
[16] Huynh, D. Q., 2009, "Metrics for 3D Rotations: Comparisons and Analysis," J. Math. Imaging Vision, **35**(2), pp. 155–164.
[17] Shiu, Y. C., and Ahmad, S., 1989, "Calibration of Wrist-Mounted Robotic Sensors by Solving Homogeneous Transform Equations of the Form $AX = XB$," IEEE Trans. Rob. Autom., **5**(1), pp. 16–29.

[18] Chou, J. C. K., and Kamel, M., 1991, "Finding the Position and Orientation of a Sensor on a Robot Manipulator Using Quaternions," Int. J. Rob. Res., **10**(3), pp. 240–254.
[19] Park, F. C., and Martin, B. J., 1994, "Robot Sensor Calibration: Solving $AX = XB$ on the Euclidean Group," IEEE Trans. Rob. Autom., **10**(5), pp. 717–721.
[20] Ackerman, M. K., and Chirikjian, G. S., 2013, "A Probabilistic Solution to the AX=XB Problem: Sensor Calibration Without Correspondence," *Geometric Science of Information*, Paris, France, Aug. 28–31.
[21] Chen, H. H., 1991, "A Screw Motion Approach to Uniqueness Analysis of Head-Eye Geometry," IEEE Conference on Computer Vision and Pattern Recognition, Maui, HI, pp. 145–151.
[22] Ackerman, M. K., Cheng, A., Shiffman, B., Boctor, E., and Chirikjian, G. S., 2013, "Sensor Calibration With Unknown Correspondence: Solving AX=XB Using Euclidean-Group Invariants," 2013 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS'13), Tokyo, Japan, Nov. 3–8, pp. 1308–1313.