Partial Bi-Invariance of SE(3) Metrics

In a flurry of articles in the mid to late 1990s, various metrics for the group of rigid-body motions, SE(3), were introduced for measuring distance between any two reference frames or rigid-body motions. During this time, it was shown that one can choose a smooth distance function that is invariant under either all left shifts or all right shifts, but not both. For example, if one defines the distance between two reference frames to be an appropriately weighted Frobenius norm of the difference of the corresponding homogeneous transformation matrices, this will be invariant under left shifts by arbitrary rigid-body motions. However, this is not the full picture—other invariance properties exist. Though the Frobenius norm is not invariant under right shifts by arbitrary rigid-body motions, for an appropriate weighting it is invariant under right shifts by pure rotations. This is also true for metrics based on the Lie-theoretic logarithm. This paper goes further to investigate the full invariance properties of distance functions on SE(3), clarifying the full subsets of motions under which both left and right invariance is possible. [DOI: 10.1115/1.4028941]

Introduction

The concept of distance between rigid-body motions (or poses) arises in a variety of applications ranging from robotics, to computer graphics, and Computer-Aided Design/Computer-Aided Manufacturing (CAD/CAM). These application include partitioning rigid-body trajectories evenly, interpolating motions, and defining cost functions in rigid-body motion based on metrics to tackle problems in attitude estimation and sensor calibration.

Given homogeneous (rigid-body) transformations of the form

\[ H_i = H(r_i, t_i) = \begin{pmatrix} R_i & t_i \\ 0 & 1 \end{pmatrix} \in \text{SE}(3) \]

a bona fide distance function, \( d : \text{SE}(3) \times \text{SE}(3) \rightarrow \mathbb{R}_{\geq 0} \) is one that satisfies the properties

\[ d(H_1, H_2) = 0 \Leftrightarrow H_1 = H_2 \]

\[ d(H_1, H_2) = d(H_2, H_1) \]

\[ d(H_1, H_2) + d(H_2, H_3) \geq d(H_1, H_3) \]

This is the general definition of a metric function, which can be defined for any space, not only SE(3). Since SE(3) is known to be a Lie group, it is desirable to consider metrics that possess additional invariance properties under simultaneous shifts of their arguments.

A number of papers over the past twenty years have introduced various metrics to define appropriate invariant concepts of distances. The study of distance functions for SE(3) originated in the mechanisms community and includes approaches based on Frobenius norms of differences of homogeneous transformations [1–4], Lie-theoretic logarithms of relative motions [5], stereographic projection into the four-sphere [6] akin to (and derived independently from) Ref. [7], metrics based on kinematic mappings [8], Lie group, it is desirable to considermetrics that possess additional invariance properties under simultaneous shifts of their arguments.

Partial Bi-Invariance

Curiously, in general it is not possible to satisfy both invariance properties simultaneously for arbitrary \( H_1, H_2, H_3 \in \text{SE}(3) \). However, for pure rotations, the situation is quite different. In this case, it is possible to define measures of distance that are invariant under both left and right shifts. That is, for arbitrary \( R_1, R_2, R_3 \in \text{SO}(3) \), it is possible to define bona fide continuous “bi-invariant” distance functions \( d_b : \text{SO}(3) \times \text{SO}(3) \rightarrow \mathbb{R}_{\geq 0} \) that do satisfy

\[ d_b(R_1R_3, R_2) = d_b(R_1R_3, R_2) \]

A simple trick can be used to create a right- or left-invariant metric from a metric with no invariance as follows. Let \( d_l(H_1, H_2) = d(1_4, H_1H_2^{-1}) \) and \( d_r(H_1, H_2) = d(1_4, H_2^{-1}H_1) \), where \( d(\cdot, \cdot) \) is a metric without invariance properties, and \( 1_4 \) is the \( 4 \times 4 \) identity matrix that serves as the group identity for SE(3).

As a side note, it is worth mentioning that the word “continuous” is critical here because without it, a discontinuous trivial metric on any group, \( G \), can be defined as

\[ d_{\text{trivial}}(g_1, g_2) = \begin{cases} \frac{1}{2} & \text{if } g_1 = g_2 \\ 0 & \text{if } g_1 \neq g_2 \end{cases} \]

and this is bi-invariant. But when \( G \) is not compact and not commutative, and is not the direct product of compact and

\[ d_{\text{trivial}}(g_1, g_2) = \begin{cases} \frac{1}{2} & \text{if } g_1 = g_2 \\ 0 & \text{if } g_1 \neq g_2 \end{cases} \]

and this is bi-invariant. But when \( G \) is not compact and not commutative, and is not the direct product of compact and
commutative groups, continuous bi-invariant metrics do not exist in general. And since SE(3) is a semidirect (rather than direct) product, there is no reason to believe that bi-invariant metrics should exist.

However, this is not the end of the story. Though it has been shown that it is not possible to define continuous distance functions for SE(3) that are bi-invariant, it is nevertheless possible to observe the following two facts:

1. There exist left-invariant metrics for SE(3) for which

\[ d_{l}(H_{1}K, H_{2}K) = d_{l}(H_{1}, H_{2}) \]

for any \( H_{1}, H_{2} \in \text{SE}(3) \) and \( K = H R_{3}, 0 \) for any \( R_{3} \in \text{SO}(3) \); and

2. Given any \( H_{1} \in \text{SE}(3) \), there exist special combinations of (not-necessarily commuting) motions \( H_{1} \) and \( H_{2} \) (which depend on \( H_{1} \)) for which the equality

\[ d_{l}(H_{1}H_{1}, H_{2}H_{1}) = d_{l}(H_{1}, H_{2}) \]

holds.

The combination of these two observations form the core of this paper, which is organized as follows. First, a detailed review of various metrics from the literature is presented. Then the partial bi-invariance properties of SE(3) metrics are derived. And finally, some ramifications of these invariances are presented.

A Survey of Metrics

This section begins with metrics for SO(3) and then moves to the SE(3) case.

The SO(3) Case. The group SO(3) is a compact connected three-dimensional matrix Lie group defined by the conditions

\[ \text{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^{T}T = I_{3}, \det R = +1 \} \]

Metrics for SO(3) can be defined from any matrix norm as

\[ d_{b}(R_{1}, R_{2}) = \| R_{1} - R_{2} \|_{N} \]

where \( N \) indicates that it is defined relative to a particular norm, \( \| \cdot \|_{N} \). For example, if the Frobenius norm is used, then

\[ d_{F}(R_{1}, R_{2}) = \| R_{1} - R_{2} \|_{F} = \sqrt{\text{tr}((R_{1} - R_{2})(R_{1} - R_{2})^{T})} \]

and from the fact that the trace is invariant under similarity transformation, it is easy to see that

\[ d_{F}(R_{1}R_{3}, R_{2}R_{3}) = \sqrt{\text{tr}(R_{3}(R_{1} - R_{2})(R_{1} - R_{2})^{T}R_{3}^{T})} = d_{F}(R_{1}, R_{2}) \]

And on the other side

\[ d_{F}(R_{1}R_{3}, R_{2}) = \sqrt{\text{tr}((R_{1} - R_{2})R_{1}R_{2}^{T})} = d_{F}(R_{1}, R_{2}) \]

This demonstrates the existence of bi-invariant metrics for SO(3), which is a property that would also hold if the induced-2-norm \( \| \cdot \|_{2} \) had been used in place of \( \| \cdot \|_{F} \). But this does not imply that every distance function on SO(3) is bi-invariant (or even invariant to shifts on one side).

For example, if instead of the usual Frobenius norm, a weighted Frobenius norm of the form

\[ d_{W}(R_{1}, R_{2}) = \| R_{1} - R_{2} \|_{W} = \sqrt{\text{tr}((R_{1} - R_{2})W(R_{1} - R_{2})^{T})} \]

is used, where \( W = W^{T} \) is positive definite but without additional structure, then the result will still be a valid distance function. But the right invariance that existed with the identity-weighted Frobenius norm will no longer hold for arbitrary right shifts (though it may for special values of \( R_{3} \) if \( W \) has special structure). Right-invariant metrics for SO(3) that lack left invariance can be constructed just as easily. And an example of a metric that is neither left- nor right invariant, consider \( d_{l}(R_{1}, R_{2}) = \| R_{1} - R_{2} \|_{1} \).

However, since SO(3) is a compact Lie group, it has a unique bi-invariant integration measure which can be used to average metrics that are not invariant, to produce ones that are invariant. Given a continuous function \( f : \text{SO}(3) \to \mathbb{R} \), there is essentially only one “correct” (bi-invariant) way to compute the integral of that function. Explicitly, if \( R \) is expressed in ZXZ Euler angles \( x, \beta, \gamma \), this is

\[
\int_{\text{SO}(3)} f(R) dR = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} f(R(x, \beta, \gamma)) \sin \beta d\beta d\beta d\gamma
\]

(Sometimes a normalizing factor of \( 1/(8\pi^{3}) \) is introduced so that \( dR = (1/(8\pi^{3})) \sin \beta d\beta d\beta d\gamma \) and \( \int_{\text{SO}(3)} 1 \, dR = 1 \), but this is only a matter of convention and not an independent choice of integration measure.)

The bi-invariance of this integration measure means that

\[
\int_{\text{SO}(3)} f(R) dR = \int_{\text{SO}(3)} f(QR) dR = \int_{\text{SO}(3)} f(RQ) dR
\]

for arbitrary fixed \( Q \in \text{SO}(3) \). And so, given a continuous left-invariant metric, it is always possible to define

\[ d_{b}(R_{1}, R_{2}) = \int_{\text{SO}(3)} d_{b}(R_{1}Q, R_{2}Q) dQ \]

(7)

This averaging on the right together with existing invariance of integration under left shifts gives a new metric that is invariant under both left and right shifts. (Averaging a metric without any invariance properties by shifting on the left and on the right, and integrating over two copies of SO(3) can also be done to create a bi-invariant metric.) The construct in Eq. (7) works because all values of \( d_{b} \) are finite (since SO(3) is compact and \( d_{b} \) is taken to be continuous) and the volume of SO(3) is finite (again from its compactness). But this argument will not work for integrating over SE(3), which is not compact.

From Euler’s theorem, it is known that any rotation matrix can be written as

\[ R(\theta, n) = I_{3} + \sin \theta N + (1 - \cos \theta)N^{2} = \exp(\theta N) \]

(8)

where \( \theta \) is the angle of rotation, and \( N = n \) is the unique skew-symmetric matrix corresponding to the unit vector defining the axis of rotation, \( n = [n_{1}, n_{2}, n_{3}]^{T} \) such that \( N x = n \times x \) for all \( x \in \mathbb{R}^{3} \). The angle can be obtained from a given \( R \) by either taking the log and then computing the Frobenius norm of the result, or by directly manipulating the above axis-angle formula to isolate \( \theta \) when it is in the range \( (0, \pi) \). The result is the same.

Given two rotations, the angle of the relative rotation between them is a measure of distance. That is,

\[ d_{\log}(R_{1}, R_{2}) = \theta(R_{1}^{T}R_{2}) = \frac{1}{2} \| \log(R_{1}^{T}R_{2}) \|_{F} \]

(9)

is a valid distance function [5] which is naturally left invariant. Moreover, since \( \log(QRQ^{T}) = (\log(R))Q^{T} \), this metric is bi-invariant.

Metrics defined using norms applied directly to the difference of two rotation matrices are sometimes referred to as being
In the case of an isotropic body, the metric in Eq. (10) also has the property that when letting $K = H(R, 0)$

$$d(KH, H_2) = d(H_1, H_2)$$

for any $R \in SO(3)$ and any $H_1, H_2 \in SE(3)$. Other metrics for $SE(3)$ that share this property of full left invariance and partial right invariance include

$$d^{(1)}_{SO(3)}(H_1, H_2) = r \cdot d_{\log}(R_1, R_2) + \| \mathbf{t}_1 - \mathbf{t}_2 \|$$

and

$$d^{(2)}_{SO(3)}(H_1, H_2) = \sqrt{r^2 \cdot d^2_{\log}(R_1, R_2) + \| \mathbf{t}_1 - \mathbf{t}_2 \|^2}$$

and

$$d^{(3)}_{SE(3)}(H_1, H_2) = \| \log(H_1^{-1}H_2) \|_W$$

where $W$ is as in Eq. (11) but for the special case of an isotropic body.

Another class of metrics is defined by shifting a unimodal probability density function of motion, $f : SE(3) \to \mathbb{R}_{>0}$, and integrating. For example,

$$d_l(H_1, H_2) = \left( \int_{SE(3)} f(H^{-1}_1 \mathbf{H}) - f(H^{-1}_2 \mathbf{H})^2 dH \right)^{1/2}$$

The integration measure for $SE(3)$ is bi-invariant, as is the $SO(3)$ integration measure. And so letting $\mathbf{H} = H^{-1}_1H$

$$d_l(H_1, H_2) = \left( \int_{SE(3)} f(H^{-1}_1 \mathbf{H}) - f(H^{-1}_2 \mathbf{H})^2 d\mathbf{H} \right)^{1/2}$$

It is often a point of confusion in the kinematics literature that $SE(3)$ can have a bi-invariant integral metric but not a bi-invariant metric. The bi-invariant integral metric for $H = H(R, t)$ is simply $d\mathbf{H} = dR dt$, where $d\mathbf{R}$ is the bi-invariant integral metric for $SO(3)$ described earlier, and $dt = dt_1 dt_2$. The proof that this is bi-invariant is given in Ref. [14]. Maybe the simplest way to say it is that conditions for the metric tensor $G(R, t)$ to be bi-invariant are more stringent than for the integration measure, which is more akin to $|det(G(R, t))|$. The former can be viewed as a $6 \times 6$ matrix function with 21 independent entries whereas the latter is a single scalar function which is invariant under shifts by unimodular matrices and arbitrary similarity transformations.

The metrics $d_l(H_1, H_2)$ do not appear to have the right invariance under $SO(3)$. But given a left-invariant metric on $SE(3)$ that does not satisfy this condition of partial right invariance, it is possible to average over rotations on the right as in Eq. (7) to form a new metric with right invariance under $SO(3)$ actions. It is not possible to average over all of $SE(3)$ because it is not compact, and the result will not be finite.

Analysis of Full Invariance

In this section, two questions are addressed in the context of $SE(3)$ metrics that are simultaneously left invariant under $SE(3)$ and right invariant under $SO(3)$. Given the discussion above, there is no loss of generality in assuming this. Both questions query special cases when $d(H_1, H_2) = d(H_1, H_2)$ where $H_3 \in SE(3)$ is not a pure rotation. Namely,

Invariance of Existing $SE(3)$ Metrics Under Left–Right $SE(3) \times SO(3)$ Actions

Note that when the body being moved is isotropic, so that for some $a \in \mathbb{R}_{>0}$ that reflects how concentrated the density $\rho$ is, and some characteristic length scale of the body, $r$, the inertia matrix will be of the form $I = arm^{-2} \cdot I_3$.

The combination of $m$ and $J$ naturally weights translations and rotations according to the properties of the body being moved without the need to artificially introduce a length parameter. A body that is very concentrated will have small $J$, and one that is very spread out will have large $J$. For example, for a point mass, $J$ will be zero, and for a long rod $J$ will be large, while $m$ will be the same in both cases. This reflects that for a long rod, the “cost” of rotation should be large, whereas rotation of a point mass around itself has no cost. In general, the metric in Eq. (10) is not right invariant.
For a pair of arbitrary homogeneous transformations, \((H_1, H_2)\), what is the largest subset of \(SE(3)\) from which right shifts \(H_1\) can be drawn and applied such that the value of the metric is unchanged?

(2) For a given right shift by an arbitrary rigid-body motion, \(H_1\), what is the largest set of pairs \((H_1, H_2)\) for which the value of the metric is unchanged?

The answers to both of these questions are addressed below.

**Answer to Question 1.** Addressing question 1 first, for given \((H_1, H_2)\), consider under what conditions \((H_1, H_2, H_3)\) can be written in the form \((H_1H_1K, H_1H_2K)\) where \(K = H(R, 0)\). Since from the discussion in the previous section, \(SE(3)\) metrics can be assumed to have left-right \(SE(3) \times SO(3)\) invariance, \(d(H_1H_1K, H_1H_2K) = d(H_1, H_2)\). And from the symmetry of a metric, this is also equal to \(d(H_1, H_1)\). Therefore, consider the two subquestions:

1a) Under what conditions on \(H_3\) is it possible to find \(H_3\) and \(K\) for which the following equations will hold simultaneously for given \((H_1, H_2)\)?

\[
H_3^{-1}H_1H_3 = H_1 \quad \text{and} \quad H_3^{-1}H_2H_3 = H_2 \quad (12)
\]

1b) Under what conditions on \(H_3\) is it possible to find \(H_3\) and \(K'\) for which the following equations will hold simultaneously for given \((H_1, H_2)\)?

\[
H_3^{-1}H_1K = H_3 \quad \text{and} \quad H_3^{-1}H_2K' = H_2 \quad (13)
\]

Computing the solution spaces for both problems will each provide branches of possible solutions, the union of which will describe the full space of \(H_3\)’s for which right invariance will hold.

Rewriting Eq. (12) so as to isolate \(H_3\) in both equations, the condition

\[
H_3^{-1}H_1H_3 = H_3 = H_3^{-1}H_2H_3K
\]

emerges, giving the condition \(H_3^{-1}H_1H_3 = H_3^{-1}H_2H_3K\). In other words, \(K\) plays no role in constraining \(H_3\). The result can further be written as

\[
(H_2H_1^{-1})H_3 = H_3H_2H_1^{-1}
\]

This is a single instance of what is known as the “\(AX = XB\) sensor calibration problem” \([17–20]\) where \(X = H_1\), and in this case

\[
A = B = H_2H_1^{-1}
\]

Depending on the specifics of \(A\) and \(B\), the solution space for \(X\) can either be empty or it can be two-, four-, or six-dimensional. When \(A = B\), solutions always exist, and when \(A = B = I_4\) then obviously \(X\) can take any value in \(SE(3)\), which is six-dimensional. The precise conditions distinguishing the remaining two cases are most easily stated by writing \(A\) (and hence \(B\)) in terms of their screw-theoretic decomposition. Let

\[
\text{screw}(n, p, \theta, d) = \begin{pmatrix} e^{i\theta} & (1 - e^{i\theta})p & +d\mathbf{n} \\ 0^T & 1 \end{pmatrix}
\]

where \(N = -N^T\) is the skew-symmetric matrix such that \(N x = n x \) for every \(x \in \mathbb{R}^3\) and \(I_3\) is the \(3 \times 3\) identity matrix. \(n\) is the direction of the axis of rotation of \(e^{i\theta}\) and \(p\) is the position of any point on the screw axis (i.e., not necessarily the usual Plücker coordinates). Then

\[
\text{screw}(n, p, \theta, d) = \text{screw}(n, p + t_n, \theta, d)
\]

for all \(t \in \mathbb{R}\). Moreover, let

\[
A = \text{screw}(n_A, p_A, \theta_A, d_A)
\]

and similarly for \(B\). Then let \(X = (R_X, t_X) \in SE(3)\). Multiplying homogeneous transformations gives

\[
X^{-1}\text{screw}(n, p, \theta, d)X = \text{screw}(R_Xn, R_Xp + t_X, \theta, d)
\]

where \(X \cdot (n, p) = (R_Xn, R_Xp + t_X)\) defines the action of \(SE(3)\) on a line defined by \((n, p)\). If \(p\) had been chosen as in Plücker coordinates, then the action \(X \cdot (n, p)\) would necessarily be more complicated to ensure that the constraint \(n \cdot p = 0\) is preserved after the action by \(X\). In the problem at hand, the line of interest is a screw axis of another rigid motion. Note that in Eq. (15) \(\theta\) and \(d\) remain unchanged under conjugation. This is a statement of the well-known fact that these are \(SE(3)\) invariants \([21,22]\). This indicates that if \(A\) and \(B\) are related as \(AX = XB\), then they must have the same invariants

\[
\theta_A = \theta_B \quad \text{and} \quad d_A = d_B
\]

This condition can be called the “compatibility of \(A\) and \(B\).”

The question then becomes, if Eq. (16) holds for given \(A\) and \(B\), what is the space of all possible \(X\)’s for which \(A = XBX^{-1}\)? The search for an appropriate \(X\) can begin by taking the \(SE(3)\) logarithm of this equation, which can be rewritten as \([14,15]\)

\[
\log^\wedge(A) = Ad(X) \log^\wedge(B)
\]

where for a general homogeneous transformation, \(H \in SE(3)\), the adjoint matrix is

\[
Ad(H(R, t)) = \begin{pmatrix} R & 0 \\ iR & R \end{pmatrix}
\]

In the case of general compatible \(A\) and \(B\), i.e., not degenerate cases in which the rotation angle\(^4\) is outside of the range \((0, \pi)\), the solution space of all possible \(X\)’s that satisfy this equation is known to be two-dimensional. This can be seen by defining

\[
\log^\wedge(A) = \begin{pmatrix} \theta_A n_A \\ v_A \end{pmatrix}
\]

Then Eq. (17) can be broken up into rotational and translational parts as

\[
n_A = R_Xn_B \quad \text{and} \quad v_A = \theta_B t_XR_Xn_B + R_Xv_B
\]

The first of these equations has a one-dimensional solution space of the form \(R_X = R(n_B, n_A)R(n_A, \phi)\), where \(\phi \in [0, 2\pi)\) is free and \(R(n_B, n_A)\) is any rotation matrix that rotates the vector \(n_B\) into \(n_A\). In particular, choose

\[
R(n_B, n_A) = I + n_B \times n_A + \frac{1 - n_B \cdot n_A}{n_B \times n_A^2}(n_B \times n_A)^2
\]

The rotation \(R(n_B, \phi)\) is given by Euler’s formula

\[^4\text{This angle is computed from the Frobenius norm } \theta_A = \| (1/2) \log R_A \| = \| (1/2) \log R_B \| = \theta_B.\]
\[ R(\mathbf{n}_b, \phi) = \mathbf{I} + \sin \phi \mathbf{n}_b \mathbf{n}_b + (1 - \cos \phi)(\mathbf{n}_b)^2 \]

Substituting \( R = R(\mathbf{n}_b, \mathbf{n}_a) \) into Eq. (20) and rearranging terms

\[ R(\mathbf{n}_b, \mathbf{n}_a)R(\mathbf{n}_b, \phi) \frac{\partial \mathbf{y}_b}{\partial \theta} = \mathbf{n}_a \mathbf{t}_x \]

(22)

The skew-symmetric matrix \( \mathbf{n}_a \) is rank 2, so a free translational degree of freedom exists in \( \mathbf{t}_x \) along the \( \mathbf{n}_a \) direction. \( \mathbf{t}_x \) can thus be described as

\[ \mathbf{t}_x = \mathbf{t}(s) = s \mathbf{m}_A + a \mathbf{m}_A + b \mathbf{m}_A \times \mathbf{n}_A \]

(23)

where \( s \in \mathbb{R} \) is a second free parameter, and \( \mathbf{m}_A \) and \( \mathbf{m}_A \times \mathbf{n}_A \) are defined to be orthogonal to \( \mathbf{n}_A \) by construction. If \( \mathbf{n}_A = [n_1, n_2, n_3]^T \) and \( n_1, n_2, n_3 \) are not simultaneously zero, then it is possible to define

\[ \mathbf{m}_A = \frac{1}{\sqrt{n_1^2 + n_2^2}} \begin{pmatrix} -n_2 \\ n_1 \\ 0 \end{pmatrix} \]

The coefficients \( a \) and \( b \) are then computed by substituting Eq. (23) into Eq. (22) and using the fact that \( \{ \mathbf{n}_A, \mathbf{m}_A, \mathbf{n}_A \times \mathbf{m}_A \} \) is an orthonormal basis for \( \mathbb{R}^3 \). Explicitly

\[ a = -\left( R(\mathbf{n}_b, \mathbf{n}_a)R(\mathbf{n}_b, \phi) \mathbf{y}_b - \mathbf{v}_A \right) \cdot (\mathbf{m}_A \times \mathbf{n}_A) \quad \text{and} \quad b = \left( R(\mathbf{n}_b, \mathbf{n}_a)R(\mathbf{n}_b, \phi) \mathbf{v}_b - \mathbf{y}_A \right) \cdot \mathbf{m}_A \]

This means that the feasible solutions can be completely parameterized as

\[ X(\phi, s) = H(R(\mathbf{n}_b, \mathbf{n}_a)R(\mathbf{n}_b, \phi), \mathbf{t}(s)) \]

(24)

where \( (\phi, s) \in [0, 2\pi) \times \mathbb{R} \) and \( H(R, \mathbf{t}) \in \text{SE}(3) \). Let us call \( X(\phi, s) \) the 2D “cylinder” of \( X \)’s that solve the problem for a single pair \((A, B)\) with \( \theta_{3A} = \theta_{3B} \in (0, \pi) \). In contrast, when \( \theta_A = \theta_B \in [0, \pi) \) the concept of logarithm used in Eq. (18) breaks down, and other methods must be used. It suffices to say that, in this case, the space of allowable \( X \)’s grows to be four-dimensional.

From this, it is clear that in the context of the present discussion there is always a solution, and for all but very special choices of \( H_1 \) and \( H_2 \), there will be two degrees of freedom in \( H_1 \).

Then, the full set of values of \( H_3 \) for which a left-invariant \( \text{SE}(3) \) metric will be right invariant can be parameterized as

\[ H_3 = H_1^{-1} H_2^0 H_1 K \]

Since \( K \) played no role in computing the freedom in \( H_1^0 \), the dimensions add. In other words, \( K \) has the three degrees of freedom of \( \text{SO}(3) \), and if \( H_1^0 \) has two additional independent degrees of freedom, then there is a whole five-dimensional subspace of \( \text{SE}(3) \) from which \( H_3 \) can be chosen for which the left-invariant metric will also be right invariant. In the degenerate case, when \( \theta_A = \theta_B \in [0, \pi) \) then the space of possible \( H_3 \)’s for which right invariance holds becomes six-dimensional, meaning that for these special values of \( H_1 \) and \( H_2 \), the metric is fully bi-invariant.

\[ \text{Answer to Question 2.} \]

The statement of question 2 uses many of the same equations, but with different variables held fixed. The question is stated as: For given arbitrary right shift \( H_1 \) what is the space of all \( H_1, H_2 \), for which the left-invariant metric remains unchanged? In analogy with question 1, this can be broken up into two subquestions:

\[ \text{(2a) For given } H_3, \text{what are all possible } H_1 \text{ and } H_2 \text{ such that } H_1^0 \text{ and } K \text{ can be found for which the following equations will hold simultaneously?} \]

\[ H_1^0 H_1 K = H_1^0 H_2 \]

(25)

\[ \text{Canceling } H_1 \text{ on the left and choosing } K = K^0 \text{ on the right} \]

\[ H_1^0 H_1 K = H_1^0 H_2 \]

Therefore, given \( H_1^0 \) it is always possible to find \( H_1^0 \), and vice versa. And so cases 1a and 1b produce exactly the same solution spaces, and do not represent different branches.

\[ \text{Answer to Question 2.} \]

The statement of question 2 uses many of the same equations, but with different variables held fixed. The question is stated as: For given arbitrary right shift \( H_1 \) what is the space of all \( H_1, H_2 \), for which the left-invariant metric remains unchanged? In analogy with question 1, this can be broken up into two subquestions:

\[ \text{(2a) For given } H_3, \text{what are all possible } H_1 \text{ and } H_2 \text{ such that } H_1^0 \text{ and } K \text{ can be found for which the following equations will hold simultaneously?} \]

\[ H_1^0 H_1 K = H_1^0 H_2 \]

(25)

\[ \text{Canceling } H_1 \text{ on the left and choosing } K = K^0 \text{ on the right} \]

\[ H_1^0 H_1 K = H_1^0 H_2 \]

Therefore, given \( H_1^0 \) it is always possible to find \( H_1^0 \), and vice versa. And so cases 1a and 1b produce exactly the same solution spaces, and do not represent different branches.
It is shown that a five-dimensional subspace of right shifts can be applied to an arbitrary pair of motions that will leave left-invariant metrics unchanged. In addition, the space of special pairs of motions for which arbitrary left and right shifts can be applied for which distance is preserved is analyzed and is found to be 11-dimensional in the general case.

Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant No. IIS-1162095. The proofreading done by Mr. Joshua Davis, Mr. Qianli Ma, and Mr. Hau Sze is greatly appreciated.

Nomenclature

\[ A \in \mathbb{R}^{n \times n} \]
\[ ||A||_F = \text{the Frobenius norm of a matrix} \]
\[ ||A||_2 = \text{an arbitrary matrix norm} \]
\[ ||W|| = \text{the weighted Frobenius norm with } W = WT \in \mathbb{R}^{n \times n} \]
\[ ||A||_{A^T} = \text{the induced 2-norm of a matrix} \]
\[ d(\cdot, \cdot) = \text{a metric/distance function} \]
\[ \text{det} = \text{the determinant of a square matrix} \]
\[ I_n = \text{the } n \times n \text{ identity matrix} \]
\[ \mathbb{R}^n = \text{n-dimensional Euclidean space} \]
\[ SE(n) = \text{the space of } n \times n \text{ matrices with real entries} \]
\[ SO(n) = \text{the group of rotations in } n\text{-dimensional Euclidean space} \]
\[ tr(A) = \text{the trace of a square matrix} \]

References


