Symmetrical Rigid Body Parameterizations For Humanoid Robots

Sipu Ruan, Jin Seob Kim, and Gregory S. Chirikjian*

***Abstract*— In the field of robotics, it is now standard to describe the motion of robots relative to a fixed world frame. For example, the position and orientation of the distal link (or robot hand) is described as a rigid-body motion relative to the robot base. However, when it comes to the description of relative rigid-body position and orientation between individual robots in a multi-robot system (e.g., a group of humanoids performing specific tasks), assessing preferred relative rigid-body position and orientation of a robot relative to another would be more appropriate. Furthermore, it would be beneficial to have a "symmetrical" parameterization in which the corresponding parameters for a pair of robots in a group are calculated in the same way for the motion and its inverse. In this paper, we discuss the possible forms of this symmetrical parameterization for the rotations and rigid body motions. This extends and updates our previously presented work. We also present the properties of the symmetrical parameterizations in terms of product formulas. Due to its "symmetrical" property, this type of the parameterization has various potential benefits in the study of humanoid robots.**

I. INTRODUCTION

The description of relative orientations and positions of robotic systems is fundamentally important in the field of robotics. For classical treatments of kinematics and comprehensive reviews of rotation parameterizations, see [1]–[6]. One of the recent trends in the robotics community includes multi-robot systems. Examples include self-reconfigurable modular robots [7], [8], collaborative multi-robot systems that can perform the repair of themselves [9], [10], and humanoid robots that can perform human activities such as soccer [11]. In particular, the well-known humanoid soccer challenge RoboCup [12], [13] aims to create a humanoid soccer team that could one day compete against a real FIFA World Cup champion. Furthermore, interactions of humanoids with humans such as communication, cooperation, and tutelage become important issues for humanoids to achieve complex tasks [14], [15]. One notable feature is that these robotic systems often consist of identical copies of unit robots to form a team or a complex assembly structure. This feature is particularly crucial in smart robots, such as a group of humanoids interacting with their environments or humans. In these cases, symmetrical description of relative position and orientation between each unit robot would be beneficial. By symmetrical description, we mean that the inversion formula in terms of the corresponding parameters involves simple symmetric mathematical operations.

For that purpose, this paper introduces possible symmetrical parameterizations on the group of 3D rotations, $SO(3)$, and rigid body motions, $SE(3)$. The main goal of the symmetrical parameterization is to parameterize rotations and rigid body motions in a "symmetrical" way, i.e., given a rotation or a rigid body motion, the corresponding inverse form looks the same way. Mathematically, given a 3×3 rotation matrix $R(\mathbf{q})$ or a 4×4 homogeneous matrix $H(\mathbf{q}')$, a symmetrical parameterization allows us to write $[R(\mathbf{q}_1)]^T = R(\mathbf{q}_2)$ and $[H(\mathbf{q}_1')]^{-1} = H(\mathbf{q}_2')$ where the corresponding relationship between $\mathbf{q}_1$ and $\mathbf{q}_2$ and between $\mathbf{q}_1'$ and $\mathbf{q}_2'$ involve simple symmetric mathematical operations.

This work builds upon the previously presented work for a symmetrical parameterization which was developed mainly for biomolecular docking application in computational structural biology [16], [17]. Once a set of parameters is obtained given a rigid body motion by using a symmetrical parameterization, there is no need to re-evaluate the parameters for its inverse motion, which facilitates the description of relative motions between unit robots. To illustrate this concept, suppose we have a group of humanoid robots to perform the activity. Relative positions and orientations between each pair should be constantly updated during the performance. A symmetrical parameterization will enhance the evaluation of relative rigid body motions because it enables each pair of humanoids to look the same way regardless of which robot is chosen as a base frame. Hence this new parameterization method has a high potential in robotics community, in particular humanoid community.

The organization of this paper is as follows. In Section II, we discuss possible forms of the symmetrical parameterizations on 3D rotations $SO(3)$ and the corresponding product formula. In the beginning, we present a brief review on the symmetrical parameterization in our earlier work, then we discuss associated and extended forms of possible symmetrical parameterizations. Furthermore, we derive the product formula of the symmetrical parameterization for $SO(3)$. Next, we move on to the discussion on the symmetrical parameterizations on the rigid body motions in the plane and in 3D space, i.e., $SE(2)$ and $SE(3)$, and the possible forms of the product formula in Section III. In order to stress the advantage of using the symmetrical parameterization, we compare the results of calculating relative motions obtained by conventional representations and the symmetrical parameterization in Section IV. Finally we conclude in Section V.

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II. Symmetrical Parameterizations on the Rotation

As is well known, 3D rotations can be described by 3×3 real matrices that satisfy the following conditions

$$RR^T = R^T R = \mathbb{I}_3 \quad \text{and} \quad \det R = +1$$

where $\mathbb{I}_3$ denotes the 3×3 identity matrix, the superscript T denotes the transpose, and "det" means the determinant of a 3×3 matrix.

A. Euler-Angles Parameterizations for Comparison

In order to better understand the concept of the "symmetrical" parameterization, let us consider one of classical parameterizations on the 3D rotation $SO(3)$, Euler-angles parameterization. The most popular choices of Euler angles are the ZXZ and ZYZ angles, which are defined as

$$R_{ZXZ}(\alpha,\beta,\gamma) = R_z(\alpha)R_x(\beta)R_z(\gamma); \tag{1}$$

$$R_{ZYZ}(\alpha,\beta,\gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma) \tag{2}$$

where $R_{x,y,z}(\theta)$ denotes the counterclockwise rotations about the x, y, and z axes of a given coordinate system by the angle θ, respectively. The ranges of Euler angles for these parameterizations are $0\le\alpha\le 2\pi$, $0\le\beta\le\pi$, and $0\le\gamma\le 2\pi$. Now let us investigate the inverse operation. Obviously, one can obtain the inverse formula as $(\alpha,\beta,\gamma)\to(-\gamma,-\beta,-\alpha)$ because

$$(R_{ZXZ}(\alpha,\beta,\gamma))^T = R_z(-\gamma)R_x(-\beta)R_z(-\alpha).$$

From this, one might think that the ZXZ Euler angles are symmetrical. However, one can readily find that the resulting angles are no longer in the correct ranges as mentioned earlier. Regarding α and γ, the situations are corrected by changing $-\gamma\to 2\pi-\gamma$ and $-\alpha\to 2\pi-\alpha$ to match the corresponding ranges. However, it is not clear how to put $-\beta$ back into the correct range. Similar issue exists in the ZYZ Euler angles parameterization. Therefore, it is clear that the Euler angles parameterizations are not "symmetrical" in regard to the corresponding inverse operation, which we seek.

B. Review on the Symmetrical Parameterization

Now, we present symmetrical parameterizations for 3D rotation. We start with a symmetrical parameterization that has been presented in engineering community [16], [17]. Before we present the forms of symmetrical parameterization, let us review an important special form of the rotation, which we call the transference rotation matrix $R(\mathbf{a},\mathbf{b})$, which is explicitly defined as [6]

$$R(\mathbf{a},\mathbf{b}) \doteq \exp\left(\theta_{ab}\cdot\frac{\widehat{\mathbf{a}\times\mathbf{b}}}{\|\mathbf{a}\times\mathbf{b}\|}\right) \tag{3}$$

$$= \mathbb{I}_3 + \widehat{\mathbf{a}\times\mathbf{b}} + \frac{(1-\mathbf{a}\cdot\mathbf{b})}{\|\mathbf{a}\times\mathbf{b}\|^2}\left(\widehat{\mathbf{a}\times\mathbf{b}}\right)^2 \tag{4}$$

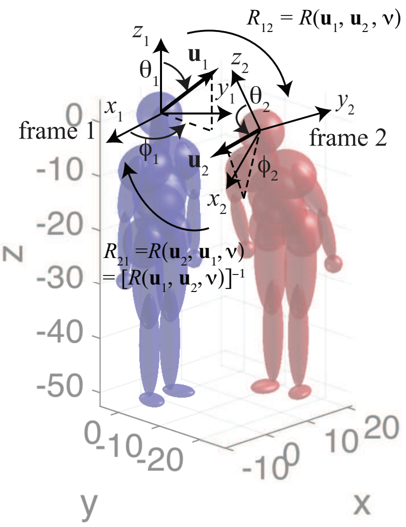

Fig. 1. Symmetrical parameterization for a 3D rotation between two humanoids

where $\mathbf{a},\mathbf{b}\in\mathbb{S}^2$ (i.e., the unit sphere in 3D space). Here $\exp(\cdot)$ represents the matrix exponential. Note that θ_{ab} should be defined as the unique angle in the range $[0,\pi]$ such that

$$\sin\theta_{ab} = \|\mathbf{a}\times\mathbf{b}\| \text{ and } \cos\theta_{ab} = \mathbf{a}\cdot\mathbf{b}.$$

This matrix transfers $\mathbf{a}$ to $\mathbf{b}$ such that $R(\mathbf{a},\mathbf{b})\mathbf{a} = \mathbf{b}$.

Some properties of this matrix $R(\mathbf{a},\mathbf{b})$ can be found as [17]

$$(R(\mathbf{a},\mathbf{b}))^{-1} = R(\mathbf{b},\mathbf{a}) = R(-\mathbf{b},-\mathbf{a}) \tag{5}$$

and

$$R(\mathbf{a},-\mathbf{b}) = R(-\mathbf{a},\mathbf{b}) \tag{6}$$

which can be shown from (4) and the elementary properties of the cross product. Another important property is

$$A\,R(\mathbf{a},\mathbf{b})A^T = R(A\mathbf{a},A\mathbf{b}) \tag{7}$$

where $A\in SO(3)$ is an arbitrary rotation, and $\mathbf{a},\mathbf{b}\in\mathbb{S}^2$ are arbitrary unit vectors. This can be shown by applying (4) and the classical kinematic equality

$$R(\mathbf{a}\times\mathbf{b}) = (R\,\mathbf{a})\times(R\,\mathbf{b}).$$

Finally, it has been proven that the following transference property holds [17]

$$R(\mathbf{a},\mathbf{b})\,\mathrm{rot}(\mathbf{a},\theta) = \mathrm{rot}(\mathbf{b},\theta)R(\mathbf{a},\mathbf{b}) \tag{8}$$

where $\mathrm{rot}(\mathbf{a},\theta)$ denotes the rotation about the direction determined by the unit vector $\mathbf{a}$ by an angle θ. In fact, this rotation is equivalent to the matrix exponential $\exp(\theta\widehat{\mathbf{a}})$. Here $\widehat{\mathbf{a}}$ denotes the 3×3 skew-symmetric matrix corresponding to a 3D vector $\mathbf{a}$. The reverse operation is defined as $(\widehat{\mathbf{a}})^\vee = \mathbf{a}$, all of which are the well known notations in the kinematics community.

Now the explicit form of a symmetrical parameterization introduced in [16], [17] is expressed as

$$R_{12}(\mathbf{u}_1,\mathbf{u}_2,\nu) \doteq \mathrm{rot}(\mathbf{u}_1,\nu/2)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,-\nu/2). \tag{9}$$

As depicted in Fig. 1, $\mathbf{u}_1$ and $\mathbf{u}_2$ can be any unit vectors in $\mathbb{S}^2$. Note that $\mathbf{u}_1$ and $\mathbf{u}_2$ are respectively described from each reference frame (frame 1 and 2), hence they should be treated as 3×1 arrays of numbers with which we can perform cross products.

Now let us consider the inverse of R_{12} as

$$\begin{aligned} R_{12}^{-1} &= [\mathrm{rot}(\mathbf{u}_2,-\nu/2)]^{-1}[R(\mathbf{u}_1,\mathbf{u}_2)]^{-1}[\mathrm{rot}(\mathbf{u}_1,\nu/2)]^{-1} \\ &= \mathrm{rot}(\mathbf{u}_2,\nu/2)R(\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_1,-\nu/2) \\ &= R_{21}. \end{aligned} \tag{10}$$

In other words, inversion operation is simply written as

$$(\mathbf{u}_1,\mathbf{u}_2,\nu)\to(\mathbf{u}_2,\mathbf{u}_1,\nu),$$

or

$$(\theta_1,\phi_1,\theta_2,\phi_2,\nu)\to(\theta_2,\phi_2,\theta_1,\phi_1,\nu)$$

in its component form, which shows that the parameterization is indeed symmetrical.

C. Possible Forms of the Symmetrical Parameterization

Note that (9) is not the only form of the symmetrical parameterization on the rotation. In this section, we list up possible forms of the symmetrical parameterization. One can find that the following forms of the parameterizations for $R(\mathbf{u}_1,\mathbf{u}_2,\nu)$ are also symmetrical:

$$\mathrm{rot}(\mathbf{u}_1,-\nu/2)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,\nu/2) \tag{11}$$
$$\mathrm{rot}(\mathbf{u}_1,\nu/2)R(\mathbf{u}_1,-\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,-\nu/2) \tag{12}$$
$$\mathrm{rot}(\mathbf{u}_1,-\nu/2)R(\mathbf{u}_1,-\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,\nu/2) \tag{13}$$
$$\mathrm{rot}(\mathbf{u}_1,\nu/2)R(-\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,-\nu/2) \tag{14}$$
$$\mathrm{rot}(\mathbf{u}_1,-\nu/2)R(-\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,\nu/2) \tag{15}$$

and

$$\mathrm{rot}(\mathbf{u}_1,-\nu/2)R(\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,\nu/2) \tag{16}$$
$$\mathrm{rot}(\mathbf{u}_1,\nu/2)R(\mathbf{u}_2,-\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,-\nu/2) \tag{17}$$
$$\mathrm{rot}(\mathbf{u}_1,-\nu/2)R(\mathbf{u}_2,-\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,\nu/2) \tag{18}$$
$$\mathrm{rot}(\mathbf{u}_1,\nu/2)R(-\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,-\nu/2) \tag{19}$$
$$\mathrm{rot}(\mathbf{u}_1,-\nu/2)R(-\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,\nu/2). \tag{20}$$

Also another possible list is obtained by switching $\mathbf{u}_1$ and $\mathbf{u}_2$ at the first and the third terms as

$$\mathrm{rot}(\mathbf{u}_2,-\nu/2)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,\nu/2) \tag{21}$$
$$\mathrm{rot}(\mathbf{u}_2,\nu/2)R(\mathbf{u}_1,-\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,-\nu/2) \tag{22}$$
$$\mathrm{rot}(\mathbf{u}_2,-\nu/2)R(\mathbf{u}_1,-\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,\nu/2) \tag{23}$$
$$\mathrm{rot}(\mathbf{u}_2,\nu/2)R(-\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,-\nu/2) \tag{24}$$
$$\mathrm{rot}(\mathbf{u}_2,-\nu/2)R(-\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,\nu/2) \tag{25}$$

and

$$\mathrm{rot}(\mathbf{u}_2,-\nu/2)R(\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_1,\nu/2) \tag{26}$$
$$\mathrm{rot}(\mathbf{u}_2,\nu/2)R(\mathbf{u}_2,-\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_1,-\nu/2) \tag{27}$$
$$\mathrm{rot}(\mathbf{u}_2,-\nu/2)R(\mathbf{u}_2,-\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_1,\nu/2) \tag{28}$$
$$\mathrm{rot}(\mathbf{u}_2,\nu/2)R(-\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_1,-\nu/2) \tag{29}$$
$$\mathrm{rot}(\mathbf{u}_2,-\nu/2)R(-\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_1,\nu/2). \tag{30}$$

Note that (11) to (20) and (21) to (30) are mutually inverse, which again shows that the list is indeed a set of the symmetrical parameterization. The above list contains two unit vectors and one angular parameter. Specifically, each contains five parameters: θ_1 and ϕ_1 from $\mathbf{u}_1$, θ_2 and ϕ_2 from $\mathbf{u}_2$, and ν. Considering that many of classical parameterizations on the rotation involve 3 parameters, we can always fix one unit vector, say $\mathbf{u}_1=\mathbf{e}_1$ ($\mathbf{e}_1=[1\,0\,0]^T$), and then describe the rotation with the remaining three parameters. If one wants to move forward to include a non-fixed $\mathbf{u}_1$, transference properties such as (7) and (8) can be used for that matter.

On the other hand, one can also define

$$\begin{aligned} R_{12} &\doteq R(\mathbf{u}_1,\mathbf{u}_2,\alpha,\beta) \\ &= \mathrm{rot}(\mathbf{u}_1,-\alpha)R(\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,\beta). \end{aligned} \tag{31}$$

When we consider its inverse, then it follows that

$$\begin{aligned} R_{12}^T &= [R(\mathbf{u}_1,\mathbf{u}_2,\alpha,\beta)]^T \\ &= [\mathrm{rot}(\mathbf{u}_2,\beta)]^T[R(\mathbf{u}_2,\mathbf{u}_1)]^T[\mathrm{rot}(\mathbf{u}_1,-\alpha)]^T \\ &= \mathrm{rot}(\mathbf{u}_2,-\beta)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,\alpha) \\ &= R(\mathbf{u}_2,\mathbf{u}_1,\beta,\alpha). \end{aligned} \tag{32}$$

This shows that (31) is also a symmetrical parameterization because the inversion formula is written as $(\mathbf{u}_1,\mathbf{u}_2,\alpha,\beta)\to(\mathbf{u}_2,\mathbf{u}_1,\beta,\alpha)$ which involves simple symmetric operation (i.e., switching the parameters). This particular form of the parameterization is useful in the product formula, which will be discussed shortly.

Again, (31) is not the only symmetrical parameterization form. By switching the role between $\mathbf{u}_1$ and $\mathbf{u}_2$ and between α and β, including sign changes, we can obtain an extensive list of possible forms. To list some of them, first by switching α and β with signs, we can see that

$$\mathrm{rot}(\mathbf{u}_1,\alpha)R(\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,-\beta) \tag{33}$$
$$\mathrm{rot}(\mathbf{u}_1,-\beta)R(\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,\alpha) \tag{34}$$
$$\mathrm{rot}(\mathbf{u}_1,\beta)R(\mathbf{u}_2,\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,-\alpha) \tag{35}$$

are all symmetrical parameterizations on 3D rotations. Also by switching the roles of $\mathbf{u}_1$ and $\mathbf{u}_2$, the additional list of possible symmetrical parameterizations are obtained such as:

$$\mathrm{rot}(\mathbf{u}_1,-\alpha)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,\beta) \tag{36}$$
$$\mathrm{rot}(\mathbf{u}_1,-\beta)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,\alpha) \tag{37}$$
$$\mathrm{rot}(\mathbf{u}_1,\beta)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_2,-\alpha) \tag{38}$$
$$\mathrm{rot}(\mathbf{u}_2,-\alpha)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,\beta) \tag{39}$$
$$\mathrm{rot}(\mathbf{u}_2,-\beta)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,\alpha) \tag{40}$$
$$\mathrm{rot}(\mathbf{u}_2,\beta)R(\mathbf{u}_1,\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,-\alpha). \tag{41}$$

D. Product Formula

With the definitions of the symmetrical parameterization, let us consider the product formula. In other words, we seek to find the corresponding symmetrical parameters $\mathbf{q}_3$ given two sets of symmetrical parameters, $\mathbf{q}_1$ and $\mathbf{q}_2$, as

$$R(\mathbf{q}_1)R(\mathbf{q}_2)=R(\mathbf{q}_3). \tag{42}$$

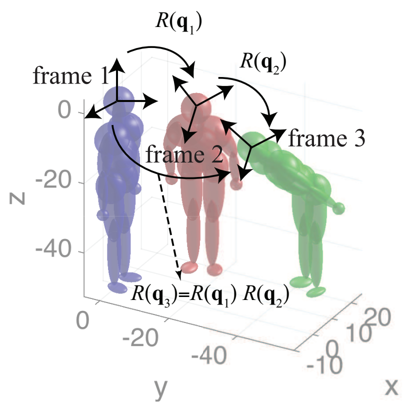

Fig. 2. A Schematic 3D rotations among three humanoids

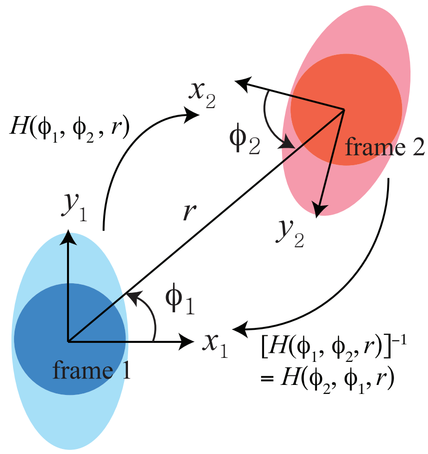

Fig. 3. Symmetrical parameterization of a planar rigid body motion between two humanoids

Fig. 2 illustrates this situation. To that end, we can consider forms such as (9) or such as (31). Since (31) is general to include (9) as a special case, let us consider the following form

$$R(\mathbf{a},\mathbf{b},\alpha,\beta)R(\mathbf{b},\mathbf{c},\beta,\gamma) = R(\mathbf{u},\mathbf{v},\vartheta,\varphi) \tag{43}$$

and seek to find $\mathbf{u},\mathbf{v},\vartheta,\varphi$. Specifically, the left hand side of (43) is written as

$$\begin{aligned}&\mathrm{rot}(\mathbf{a},-\alpha)R(\mathbf{b},\mathbf{a})\mathrm{rot}(\mathbf{b},\beta)\,\mathrm{rot}(\mathbf{b},-\beta)R(\mathbf{c},\mathbf{b})\mathrm{rot}(\mathbf{c},\gamma)\\ &= \mathrm{rot}(\mathbf{a},-\alpha)R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b})\mathrm{rot}(\mathbf{c},\gamma)\end{aligned}$$

One might think that $R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b}) = R(\mathbf{c},\mathbf{a})$, but this is not true. In general this can be expressed as

$$R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b}) = \mathrm{rot}(\mathbf{a},\theta_1)R(\mathbf{c},\mathbf{a})\mathrm{rot}(\mathbf{c},\theta_2) \tag{44}$$

where θ_1 and θ_2 are functions of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

Now when we apply (8), then we obtain

$$R(\mathbf{c},\mathbf{a})\mathrm{rot}(\mathbf{c},\theta_2) = \mathrm{rot}(\mathbf{a},\theta_2)R(\mathbf{c},\mathbf{a}) \tag{45}$$

and

$$\mathrm{rot}(\mathbf{a},\theta_1)R(\mathbf{c},\mathbf{a}) = R(\mathbf{c},\mathbf{a})\mathrm{rot}(\mathbf{c},\theta_1). \tag{46}$$

Then (44) can be written as the forms

$$R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b}) = \mathrm{rot}(\mathbf{a},(\theta_1+\theta_2))R(\mathbf{c},\mathbf{a}) \tag{47}$$

and

$$R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b}) = R(\mathbf{c},\mathbf{a})\mathrm{rot}(\mathbf{c},(\theta_1+\theta_2)). \tag{48}$$

Since we do not need to distinctively use θ_1 and θ_2, let us define $\theta \doteq \theta_1+\theta_2$. Then we have the following simple form

$$R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b}) = \mathrm{rot}(\mathbf{a},\theta)R(\mathbf{c},\mathbf{a}) \tag{49}$$

which, in this case, θ can be obtained numerically by basic matrix operations. The result of θ is

$$\theta = \|\log^{\vee}\left(R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b})R^T(\mathbf{c},\mathbf{a})\right)\| \tag{50}$$

where $\vee$ operation is defined previously, and log of a rotation matrix is well defined [6].

After evaluating the value of θ, the expression of the product formula can finally be obtained as

$$\begin{aligned}(43) &= \mathrm{rot}(\mathbf{a},-\alpha)R(\mathbf{b},\mathbf{a})\mathrm{rot}(\mathbf{b},\beta)\mathrm{rot}(\mathbf{b},-\beta)R(\mathbf{c},\mathbf{b})\mathrm{rot}(\mathbf{c},\gamma)\\ &= \mathrm{rot}(\mathbf{a},-\alpha)R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b})\mathrm{rot}(\mathbf{c},\gamma)\\ &= \mathrm{rot}(\mathbf{a},-\alpha)\mathrm{rot}(\mathbf{a},\theta)R(\mathbf{c},\mathbf{a})\mathrm{rot}(\mathbf{c},\gamma)\\ &= \mathrm{rot}(\mathbf{a},-(\alpha-\theta))R(\mathbf{c},\mathbf{a})\mathrm{rot}(\mathbf{c},\gamma)\\ &= R(\mathbf{a},\mathbf{c},\alpha-\theta,\gamma),\end{aligned} \tag{51}$$

thus, $\mathbf{u}=\mathbf{a}$, $\mathbf{v}=\mathbf{c}$, $\vartheta=\alpha-\theta$, $\varphi=\gamma$.

III. Symmetrical Parameterization on Rigid Body Motions

In this section, we discuss a symmetrical parameterization on rigid body motions, in the plane $SE(2)$ and in 3D space $SE(3)$.

A. *Symmetrical Parameterization on $SE(2)$*

A symmetrical parameterization of planar rigid body motions was presented in [16], [17]. Here we present a brief review on the definition.

Suppose there are two Humanoids in the plane (see Fig. 3). The homogeneous transformation matrix for a planar rigid body motion with the symmetrical parameterization is

$$H(\phi_1,\phi_2,r) = \begin{pmatrix} -\cos(\phi_1-\phi_2) & \sin(\phi_1-\phi_2) & r\cos\phi_1 \\ -\sin(\phi_1-\phi_2) & -\cos(\phi_1-\phi_2) & r\sin\phi_1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{52}$$

Each parameter has the following interpretation. First, r is the radial distance between the origins of two reference frames. And ϕ_i $(i=1,2)$ denotes the polar-coordinate angle in each coordinate system. Note that the relative orientation of frame 2 relative to frame 1 is computed as $\theta = \pi+\phi_1-\phi_2$. This renders θ to not appear in $H(\phi_1,\phi_2,r)$.

When we compute the inverse of $H(\phi_1,\phi_2,r)$, then first one can see that the rotation part becomes

$$\begin{pmatrix} -\cos(\phi_1-\phi_2) & -\sin(\phi_1-\phi_2) \\ \sin(\phi_1-\phi_2) & -\cos(\phi_1-\phi_2) \end{pmatrix}$$
$$= \begin{pmatrix} -\cos(\phi_2-\phi_1) & \sin(\phi_2-\phi_1) \\ -\sin(\phi_2-\phi_1) & -\cos(\phi_2-\phi_1) \end{pmatrix},$$

and the translation part is computed as

$$-r\cos\phi_1(-\cos(\phi_1-\phi_2)) - r\sin\phi_1(-\sin(\phi_1-\phi_2)) = r\cos\phi_2$$
$$-r\cos\phi_1(\sin(\phi_1-\phi_2)) - r\sin\phi_1(-\cos(\phi_1-\phi_2)) = r\sin\phi_2$$

hence

$$[H(\phi_1,\phi_2,r)]^{-1} = \begin{pmatrix} -\cos(\phi_2-\phi_1) & \sin(\phi_2-\phi_1) & r\cos\phi_2 \\ -\sin(\phi_2-\phi_1) & -\cos(\phi_2-\phi_1) & r\sin\phi_2 \\ 0 & 0 & 1 \end{pmatrix} = H(\phi_2,\phi_1,r)$$

which proves that this parameterization is indeed symmetrical.

B. Symmetrical Parameterization on $SE(3)$

Let us consider two reference frames attached to each head of the humanoids in the similar situation as in Fig. 1. The position and orientation of the second humanoid (frame 2) viewed from the first one (frame 1) is written as $g_{12} = (R_{12}, r_{12}\mathbf{u}_1) \in SE(3)$, which can be expressed by the homogenous transformation matrix as

$$H(g_{12}) = H(R_{12}, r_{12}\mathbf{u}_1) = \begin{pmatrix} R_{12} & r_{12}\mathbf{u}_1 \\ \mathbf{0}^T & 1 \end{pmatrix}$$

where the translation vector connecting the origins is described by the spherical coordinates (i.e., $\mathbf{r} = r\mathbf{u}$ with $r \in \mathbb{R}_{\geq 0}$ and $\mathbf{u} \in \mathbb{S}^2$).

First, we seek a symmetrical parameterization of the form

$$H(\mathbf{u}_1,\mathbf{u}_2,\nu,r_{12}) = H(\theta_1,\phi_1,\theta_2,\phi_2,\nu,r_{12}) = \begin{pmatrix} R(\mathbf{u}(\theta_1,\phi_1),\mathbf{u}(\theta_2,\phi_2),\nu) & r_{12}\mathbf{u}(\theta_1,\phi_1) \\ \mathbf{0}^T & 1 \end{pmatrix} \quad (53)$$

where $\mathbf{u}_i = \mathbf{u}(\theta_i,\phi_i)$ $(i=1,2)$. In other words, a spherical-coordinate parameterization is used for the unit vectors. Compared with $SO(3)$ case, we need only one additional parameter r_{12} which is the distance between the origins of frame 1 and 2, because the information on the direction of the translation is encoded in $\mathbf{u}_1$ and $\mathbf{u}_2$.

On the other hand, the position and orientation of frame 1 (the first humanoid) in the frame 2 attached to the second humanoid becomes $g_{21} = (R_{21}, r_{12}\mathbf{u}_2)$. Then by using the fact that $g_{12} \circ g_{21} = g_{21} \circ g_{12} = e$, it follows that

$$g_{21} = g_{12}^{-1} = (R_{12}^T, -r_{12}R_{12}^T\mathbf{u}_1).$$

Note that the vectors $\mathbf{u}_i$ $(i=1,2)$ lie on the same line connecting the origins of frame 1 and 2, although they are defined in their respective reference frames and their directions are opposite. In other words, the unit vector $\mathbf{u}_1$ is no longer an arbitrary fixed vector, unlike as in the case

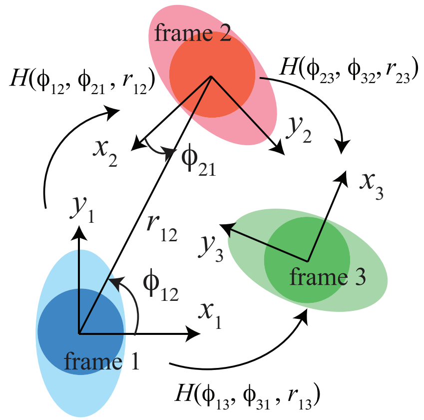

Fig. 4. Planar rigid body motions among three humanoids

of 3D rotation. Due to this fact, not all the symmetrical parameterizations listed in the previous section hold for (53). Specifically we choose

$$R(\mathbf{u}_1,\mathbf{u}_2,\nu) \doteq \mathrm{rot}(\mathbf{u}_1,\pm\nu/2)R(\mathbf{u}_2,-\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,\mp\nu/2). \quad (54)$$

Then by using (5) and (10), we obtain

$$\begin{aligned} -rR_{12}^T\mathbf{u}_1 &= -r\,\mathrm{rot}(\mathbf{u}_2,\pm\nu/2)R(\mathbf{u}_1,-\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,\mp\nu/2)\mathbf{u}_1 \\ &= -r\,\mathrm{rot}(\mathbf{u}_2,\pm\nu/2)R(\mathbf{u}_1,-\mathbf{u}_2)\mathbf{u}_1 \\ &= -r\,\mathrm{rot}(\mathbf{u}_2,\pm\nu/2)(-\mathbf{u}_2) \\ &= r\mathbf{u}_2. \end{aligned} \quad (55)$$

Hence the inversion formula for these parameters is written as

$$(\mathbf{u}_1,\mathbf{u}_2,\nu,r_{12}) \to (\mathbf{u}_2,\mathbf{u}_1,\nu,r_{12}), \quad (56)$$

or in the component form,

$$(\theta_1,\phi_1,\theta_2,\phi_2,\nu,r_{12}) \to (\theta_2,\phi_2,\theta_1,\phi_1,\nu,r_{12}).$$

Also when we choose

$$R(\mathbf{u}_1,\mathbf{u}_2,\alpha,\beta) \doteq \mathrm{rot}(\mathbf{u}_1,\pm\alpha)R(\mathbf{u}_2,-\mathbf{u}_1)\mathrm{rot}(\mathbf{u}_2,\mp\beta) \quad (57)$$

then we have

$$\begin{aligned} -rR_{12}^T\mathbf{u}_1 &= -r\,\mathrm{rot}(\mathbf{u}_2,\pm\beta)R(\mathbf{u}_1,-\mathbf{u}_2)\mathrm{rot}(\mathbf{u}_1,\mp\alpha)\mathbf{u}_1 \\ &= -r\,\mathrm{rot}(\mathbf{u}_2,\pm\beta)R(\mathbf{u}_1,-\mathbf{u}_2)\mathbf{u}_1 \\ &= -r\,\mathrm{rot}(\mathbf{u}_2,\ \pm\beta)(-\mathbf{u}_2) \\ &= r\mathbf{u}_2. \end{aligned} \quad (58)$$

Hence in this case, a symmetrical parameterization takes the form as

$$H(\mathbf{u}_1,\mathbf{u}_2,\alpha,\beta,r_{12}) = H(\theta_1,\phi_1,\theta_2,\phi_2,\alpha,\beta,r_{12}) = \begin{pmatrix} R(\mathbf{u}(\theta_1,\phi_1),\mathbf{u}(\theta_2,\phi_2),\alpha,\beta) & r_{12}\mathbf{u}(\theta_1,\phi_1) \\ \mathbf{0}^T & 1 \end{pmatrix}. \quad (59)$$

In total, symmetrical parameterizations such as (53) or (59) are indeed symmetrical for the rigid body motions.

C. Product Formula for $SE(2)$

Given

$$H(\phi_{13},\phi_{31},r_{13}) = H(\phi_{12},\phi_{21},r_{12})H(\phi_{23},\phi_{32},r_{23}),$$

we want to obtain the corresponding symmetrical parameters (see Fig. 4). First the left-hand side is written as

$$H(\phi_{13},\phi_{31},r_{13}) = \begin{pmatrix} -\cos(\phi_{13}-\phi_{31}) & \sin(\phi_{13}-\phi_{31}) & r_{13}\cos\phi_{13} \\ -\sin(\phi_{13}-\phi_{31}) & -\cos(\phi_{13}-\phi_{31}) & r_{13}\sin\phi_{13} \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, the right-hand side is computed as follows. First the corresponding rotation part is computed as

$$\begin{pmatrix} \cos(\phi_{12}-\phi_{21}+\phi_{23}-\phi_{32}) & -\sin(\phi_{12}-\phi_{21}+\phi_{23}-\phi_{32}) \\ \sin(\phi_{12}-\phi_{21}+\phi_{23}-\phi_{32}) & \cos(\phi_{12}-\phi_{21}+\phi_{23}-\phi_{32}) \end{pmatrix}$$

and the translation part is as

$$\begin{pmatrix} r_{12}\cos\phi_{12} - r_{23}\cos(\phi_{12}+\phi_{23}-\phi_{21}) \\ r_{12}\sin\phi_{12} - r_{23}\sin(\phi_{12}+\phi_{23}-\phi_{21}) \end{pmatrix}.$$

From considering the magnitude of the translational part, we obtain the following radial parameter as

$$r_{13} = \sqrt{r_{12}^2 + r_{23}^2 - 2r_{12}r_{23}\cos(\phi_{21}-\phi_{23})}. \tag{60}$$

Also, from considering

$$r_{13}\cos\phi_{13} = r_{12}\cos\phi_{12} - r_{23}\cos(\phi_{12}+\phi_{23}-\phi_{21});$$
$$r_{13}\sin\phi_{13} = r_{12}\sin\phi_{12} - r_{23}\sin(\phi_{12}+\phi_{23}-\phi_{21}),$$

we can obtain ϕ_{13} as

$$\phi_{13} = \text{Atan2}(x_{13}, y_{13}) \tag{61}$$

where

$$x_{13} = r_{12}\cos\phi_{12} - r_{23}\cos(\phi_{12}+\phi_{23}-\phi_{21})$$
$$y_{13} = r_{12}\sin\phi_{12} - r_{23}\sin(\phi_{12}+\phi_{23}-\phi_{21}).$$

Finally by solving

$$\cos(\pi+\phi_{13}-\phi_{31}) = \cos(\phi_{12}-\phi_{21}+\phi_{23}-\phi_{32})$$
$$\sin(\pi+\phi_{13}-\phi_{31}) = \sin(\phi_{12}-\phi_{21}+\phi_{23}-\phi_{32})$$

we obtain

$$\phi_{31} = \pi + \phi_{13} - (\phi_{12}-\phi_{21}+\phi_{23}-\phi_{32}). \tag{62}$$

In other words, (60), (61), and (62) constitute the product formula for $SE(2)$.

Note that, unlike $SE(2)$, we do not have a simple form of the product formula for $SE(3)$ corresponding to (54) and (57). One can resort to the inverse kinematics procedure as in [17] instead.

IV. Comparisons Between Conventional and Symmetrical Parameterization

The benefits of the symmetrical parameterization lie on its "symmetry" properties in conducting inverse operations and simple expressions in calculations. In this section, we briefly compare the proposed symmetrical parameterization with conventional one for rigid body motions, in particular, by deriving product formulas in both $SO(3)$ and $SE(2)$ cases.

A. Comparison of Product Formulas on $SO(3)$

We first calculate the product formula for pure rotations using Euler angle conventions. Let us consider three relative frames A, B and C with fixed origins and calculate the relative rotations using the ZYZ Euler angles. Let $(\alpha_1,\beta_1,\gamma_1)$ and $(\alpha_2,\beta_2,\gamma_2)$ be the Euler angles describing rotations between frames A and B and frames B and C respectively. Then, their relative rotations become

$$R_{AB}(\alpha_1,\beta_1,\gamma_1) = R_z(\alpha_1)R_y(\beta_1)R_z(\gamma_1)$$
$$R_{BC}(\alpha_2,\beta_2,\gamma_2) = R_z(\alpha_2)R_y(\beta_2)R_z(\gamma_2)$$

Applying matrix product, we get the relative rotation between frames A and C, which can be shown as

$$R_{AC}(\alpha,\beta,\gamma) = R_z(\alpha_1)R_y(\beta_1)R_z(\gamma_1)R_z(\alpha_2)R_y(\beta_2)R_z(\gamma_2) \tag{63}$$

To get the the corresponding (α,β,γ) set that describes the relative rotation between frames A and C, we have to figure out the explicit matrix expression for (63), and extract the angles from it, which takes effort to do. Another inconvenience for this procedure is that, for different Euler angles representations, there are different expressions to extract angles from the matrix. However, the symmetrical parameterization saves us from extracting angles from the complicated matrix, instead, gives a much simpler result directly related to the input parameters.

Let $(\mathbf{a},\mathbf{b},\alpha,\beta)$ and $(\mathbf{b},\mathbf{c},\beta,\gamma)$ be parameters that describe relative rotations between frames A and B, and frames B and C respectively. From (51), the relative rotations between frames A and C can be parameterized as $(\mathbf{a},\mathbf{c},\alpha-\theta,\gamma)$, where $\theta = \|\log^\vee\left(R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b})R^T(\mathbf{c},\mathbf{a})\right)\|$, which can be computed efficiently.

Table I illustrates the comparisons of product formulas between conventional and symmetrical parameterizations.

TABLE I
COMPARISONS OF PRODUCT FORMULAS BETWEEN CONVENTIONAL AND SYMMETRICAL PARAMETERIZATIONS ON $SO(3)$

Method	Symmetrical	Conventional (ZYZ Euler angles)
Parameters	$\mathbf{a},\mathbf{b},\mathbf{c},\alpha,\beta,\gamma$	$\alpha_1,\beta_1,\gamma_1,\alpha_2,\beta_2,\gamma_2$
Product Formula Representations	$R(\mathbf{u},\mathbf{v},\vartheta,\varphi)$	$R(\alpha,\beta,\gamma)$
Results	$\mathbf{u}=\mathbf{a}$; $\mathbf{v}=\mathbf{c}$; $\vartheta=\alpha-\theta$ $\varphi=\gamma$	$\alpha=\text{Atan2}(r_{13},r_{23})$ $\beta=\cos^{-1}(r_{33})$ $\gamma=\text{Atan2}(-r_{31},r_{32})$

Note: $\theta = \|\log^\vee\left(R(\mathbf{b},\mathbf{a})R(\mathbf{c},\mathbf{b})R^T(\mathbf{c},\mathbf{a})\right)\|$,
and r_{ij} is the i,j-th element in the resulting rotation matrix.

B. Comparison of Product Formulas on $SE(2)$

We now compare the product formulas on $SE(2)$ by first applying the conventional method. In polar coordinates, let $(\theta_{12},\phi_{12},r_{12})$ and $(\theta_{23},\phi_{23},r_{23})$ denote the relative rigid body motions between frames A and B, and frames B and C respectively, where (r_{ij},θ_{ij}) pair describes the relative position and ϕ_{ij} represents the relative rotation between the two frames. Written in homogeneous matrix, the product can be calculated as

$$H(\theta_{13},\phi_{13},r_{13}) = H(\theta_{12},\phi_{12},r_{12})H(\theta_{23},\phi_{23},r_{23})$$
$$= \begin{pmatrix} \cos\phi_{13} & -\sin\phi_{13} & x_{13} \\ \sin\phi_{13} & \cos\phi_{13} & y_{13} \\ 0 & 0 & 1 \end{pmatrix}$$

where,

$$\begin{aligned} x_{13} &= r_{23}\cos\theta_{23}\cos\phi_{12} - r_{23}\sin\theta_{23}\sin\phi_{12} + r_{12}\cos\theta_{12} \\ y_{13} &= r_{23}\cos\theta_{23}\sin\phi_{12} + r_{23}\sin\theta_{23}\cos\phi_{12} + r_{12}\sin\theta_{12} \\ \phi_{13} &= \phi_{12} + \phi_{23} \end{aligned} \tag{64}$$

Considering the magnitude of the translation part, the radial parameter can be shown as

$$r_{13} = \sqrt{x_{13}^2 + y_{13}^2} \tag{65}$$

Also, the angle for polar coordinates θ_{13} can be obtained as

$$\theta_{13} = \text{Atan2}(x_{13}, y_{13}) \tag{66}$$

Equations (64), (65) and (66) constitute the product formula by the conventional representation. Compared with these, product formulas (60), (61) and (62), obtained by symmetrical parameterization, give simpler expressions in calculation.

Moreover, symmetrical parameterization has more advantage in inverse expressions, that is, the relative motions of frame A with respect to frame C. In conventional form, although r_{31} and ϕ_{31} are easy to get, for θ_{31}, we have to calculate the matrix inverse of $H(\theta_{13},\phi_{13},r_{13})$, where

$$H(\theta_{31},\phi_{31},r_{31}) = H^{-1}(\theta_{13},\phi_{13},r_{13})$$
$$= \begin{pmatrix} R(\phi_{13}) & \mathbf{t} \\ 0^T & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} R^T(\phi_{13}) & -R^T(\phi_{13})\mathbf{t} \\ 0^T & 1 \end{pmatrix}$$

then, extract θ_{31} from the translation part.

However, by applying the symmetrical parameterization, the only operation we need to do is to swap the positions of ϕ_{13} and ϕ_{31}, and get the corresponding inverse $H(\phi_{31},\phi_{13},r_{31})$, since ϕ_{13} and ϕ_{31} are already defined as viewed in frame A and frame C respectively, and $r_{13} = r_{31}$ is just the distance between the two frame origins.

V. Conclusions

We presented and extended the concept of the symmetrical parametrization introduced earlier for biomolecular docking, where parameters are presented in a symmetrical manner under the inversion. In particular, we investigated possible forms of the symmetrical parameterizations on $SO(3)$. We derived the product formula for these symmetrical parameterizations on the rotation, which is important in describing mutual relations among rigid bodies. We also investigated the possible forms of the symmetrical parameterization on the rigid body motions in the plane and 3D space, and the product formula for planar rigid body motions. In the end, we compared the expressions of product formulas using conventional and symmetrical parameterization, showing that the results are much simpler and more convenient to get by symmetrical parameterization. Due to the symmetrical property, this type of parameterization will greatly facilitate the description of relative motions between robots that together perform complex tasks, which emphasizes a high potential in the application of humanoid robots.

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