Bayesian Filtering for Orientational Distributions: A Fourier Approach

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***Abstract*—A Bayesian filter for rotation groups in 2D and 3D is derived. The prior, propagator, and measurement probability densities are all assumed to be bandlimited functions on $SO(2)$ or $SO(3)$, expressed as a Fourier series on these compact Lie groups. The posterior, which has a higher bandlimit, is computed and then low-pass filtered, resulting in a bandlimited approximation. The benefits and drawbacks of the Fourier approach presented here are examined in contrast to the Gaussian approach designed for small error covariances. While the Gaussian approach is much faster, it breaks down for large error covariances. The point where the Gaussian approach breaks down is analyzed with the Fourier method, indicating the range of error sizes where the switch to Fourier methods is required.**

I. Introduction

Recently in the literature, the topic of estimation on the rotation group, $SO(3)$, has received considerable attention [1]–[14]. Applications include error propagation in robotic manipulators [15]–[17], localization in mobile robotics [18]–[20], and spacecraft attitude estimation [21]–[25]. Such topics have a very long history, including the pioneering efforts in the controls community in [26]–[30], and in the mathematics community even earlier [31], [32].

This paper is concerned with developing closed-form formulas to propagate uncertainty (via the noncommutative convolution theorem) and to fuse priors and measurement distributions in closed form in the case of continuous rather than discrete time. There has been a body of work that are related to the directional estimation of positions on the unit circle and sphere using Fourier density functions [33]–[37]. In this paper, our main interest lies in the estimation on the orientation, i.e., rotation group $SO(2)$ and $SO(3)$. Given a rotational system, the posterior probability density function (pdf) is computed as the product of prior and measurement distributions. We present the method based on the Fourier approach to compute this posterior distribution on the rotation group in the case of large errors. We consider Gaussian functions as an example, which are typical as likelihood functions of the orientation. The remainder of the paper is structured as follows. Section II provides a brief review of engineering-based filtering methods on Lie groups under the assumption of small error covariances. The main purpose of that section is to illustrate both the convenience and limitations of the assumption of small errors.

Section III explains how propagation of large orientation errors on the rotation group can be implemented efficiently using noncommutative harmonic analysis. Section IV explains how Bayesian fusion can be implemented as a Fourier calculation. Section V then illustrates these ideas with numerical calculations.

As a matter of notational convenience, $\exp : so(3) \to SO(3)$ is the exponential map. Explicitly, if $X = -X^T \in \mathbb{R}^{3\times 3}$ then we say that this is an element of $so(3)$. The result of exponentiating this matrix is a rotation, $R = \exp(X) \in SO(3)$. In other words, $RR^T = R^TR = \mathbb{I}_3$ and $\det R = +1$, where $\mathbb{I}_3$ denotes a 3×3 identity matrix. Going the other way, the logarithm of any rotation matrix with rotation angle $0 \le \theta < \pi$ can be defined as $\log(R)$. In probabilistic settings the set of measure zero defined by $\theta = \pi$ is unimportant. To distinguish between the matrix exponential and the scalar exponential, we refer to the latter as e^x. Both concepts will appear in our formulation.

II. Alternatives: Gaussians for Small Errors

When uncertainties are small both in the prior and measured distributions a Gaussian, with covariance $\Sigma = [\sigma_{ij}]$, of the form

$$f_\Sigma(R) \doteq \frac{1}{(2\pi)^{\frac{3}{2}}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}[\log^\vee(R)]^T \Sigma^{-1} \log^\vee(R)} \tag{1}$$

is a convenient way to describe the distribution of possible rotations where $\vee$ converts 3×3 skew symmetric matrices (or elements of $so(3)$) to three-vectors. That is, if $X = -X^T$ then $X\mathbf{v} = \mathbf{x} \times \mathbf{v}$ where $\mathbf{x} = X^\vee$ for any $\mathbf{v} \in \mathbb{R}^3$. Here

$$X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} = \sum_{k=1}^{3} x_i E_i$$

and $E_i^\vee = \mathbf{e}_i$, the i^{th} natural basis vector for $\mathbb{R}^3$.

The Gaussian in (1) is centered at the identity, and one centered at R_μ is obtained by shifting it and evaluating $f_\Sigma(R_\mu^{-1}R)$.

If $R = \exp(X)$ is the exponential parameterization for $SO(3)$ as computed in [38], then the Haar measure for $SO(3)$ can be computed as $dR = |J(\mathbf{x})|d\mathbf{x}$ where $d\mathbf{x} = dx_1dx_2dx_3$ and the Jacobian determinant in this parameterization has the property that $|J(\mathbf{x})| = 1 + O(\|\mathbf{x}\|^2)$.

The significance of this is that for small values of $\|\mathbf{x}\|$, integrating over $SO(3)$ is very much like integrating over $\mathbb{R}^3$. For example, the covariance can be defined as

$$\Sigma \doteq \int_{SO(3)} \log^\vee(R)[\log^\vee(R)]^T f_\Sigma(R) dR.$$

In exponential coordinates $\log^\vee(\exp X) = \mathbf{x} \in B_\pi$ (the solid ball in $\mathbb{R}^3$ of radius π centered at $\mathbf{x} = \mathbf{0}$) and

$$\Sigma = \int_{B_\pi} \mathbf{x}\mathbf{x}^T f_\Sigma(\exp X) |J(\mathbf{x})| d\mathbf{x}.$$

Here $|\cdot|$ and $\|\cdot\|$ respectively denote the determinant of a matrix and the Euclidean norm of a vector or a matrix. For concentrated probability density functions, $\|\Sigma\| \ll 1$, which means that the tails of $f_\Sigma(\exp X)$ decay to negligible levels well before $|J(\mathbf{x})|$ deviates substantially from 1, and before the tails travel outside of B_π. As a result, for concentrated pdfs, the changes $|J(\mathbf{x})| \to 1$ and $B_\pi \to \mathbb{R}^3$ can be made to simplify the computation of the integral. In fact, this is the justification for the definition in (1), which breaks down as $\|\Sigma\|$ grows to larger values.

Errors of Markovian random processes propagate by convolution. Also, it has been shown that, for small errors, the mean and covariance of the convolution of $f_{\Sigma_1}(R_{\mu_1}^{-1}R)$ and $f_{\Sigma_2}(R_{\mu_2}^{-1}R)$ are given by [15], [17], [32]

$$\begin{aligned} R_{\mu_{1*2}} &= R_{\mu_1} R_{\mu_2}, \\ \Sigma_{1*2} &= Ad^{-1}(R_{\mu_2})\Sigma_1 Ad^{-T}(R_{\mu_2}) + \Sigma_2 \end{aligned} \tag{2}$$

where Ad denotes the adjoint matrix. These equations are general, and hold for any unimodular Lie group, as long as errors are small. This is a broad class including $SO(3)$. In the special case of $SO(3)$, which is the focus of this paper, the adjoint matrix is defined as

$$Ad(R) = R. \tag{3}$$

Moreover, given a noisy kinematic system of the form

$$\left(R^{-1}dR\right)^\vee = \mathbf{a}\, dt + B\, d\mathbf{W} \tag{4}$$

(where $d\mathbf{W}$ are increments of Wiener process which is uncorrelated white noise of unit strength), then when $\mathbf{a}$ and B are constant $\|BB^T\| \ll 1$, the ensemble solutions with initial conditions $R(0)$ will have mean and covariance given by [19]

$$R_\mu(t) = R(0)\, \exp\left(t \sum_{i=1}^{3} E_i a_i\right)$$

and

$$\Sigma(t) = Ad^{-1}(R_\mu(t))\, \Sigma'(t)\, Ad^{-T}(R_\mu(t)) \tag{5}$$

where

$$\Sigma'(t) = \int_0^t Ad^{-1}(R_\mu(\tau)) BB^T Ad^{-T}(R_\mu(\tau)) d\tau.$$

And in many instances of practical interest, these integrals can be computed in closed form both for the case of $SO(3)$ discussed here, and for the group of Euclidean motions, $SE(3)$, discussed in [39].

Moreover, Bayesian fusion methods have been developed for expressing the product of two concentrated Gaussians on $SO(3)$ of the form $f_{\Sigma_1}(R_{\mu_1}^{-1}R)$ and $f_{\Sigma_2}(R_{\mu_2}^{-1}R)$ in a result that is of the same form [40]. Therefore, all of the parts for an effective filter on $SO(3)$ have been put in place, as explained in [8]. When measurement and propagation errors are relatively small, the above framework is appealing as an invariant filter. Perhaps this is why several groups have independently, and nearly in parallel, developed similar ideas. However, difficulties arise when very large errors are present. Some authors attempt to redefine the concept of a Gaussian on $SO(3)$ by dividing by $|J(\mathbf{x})|$ when the tails are heavy, since this quantity can no longer be assumed to be equal to unity. Others seek to somehow fold the Gaussian around the ball B_π or around a maximal torus in $SO(3)$. While such ideas are not without merit, particularly when $\Sigma = \sigma^2\mathbb{I}$, they lead to complications such as loss of simple propagation formulas in (2) and (5). And more importantly, form closure cannot be assumed in the sense that a Gaussian defined by dividing and/or folding will not be guaranteed to be closed under convolution when $\|\Sigma\|$ is not sufficiently small.

This paper therefore examines a completely different alternative paradigm based on Fourier analysis on $SO(3)$, as described in the next section. In the Fourier approach, distributions with heavy tails are no problem. In fact, the more spread out a distribution is, the easier it is to handle.

III. Error Propagation Using Fourier Analysis

Given the estimate of a rotating system at time t is $f_t(R)$, and if the transition probability density describing how any R will diffuse at the next instant in time is $f_{\Delta t}(R)$ then in the absence of any measurements, then under the Markov assumption, the estimate at time $t + \Delta t$ will be

$$(f_t * f_{\Delta t})(R) \doteq \int_{SO(3)} f_t(A) f_{\Delta t}(A^{-1}R) dA \tag{6}$$

where $f_{\Delta t}(A^{-1}R)$ can be interpreted as the probability density corresponding to R at time Δt given A at time 0. The Fourier coefficients for any square-integrable function on $SO(3)$ are defined as

$$\hat{f}^l_{nm} = \int_{SO(3)} f(R) \overline{U^l_{mn}(R)} dR. \tag{7}$$

The set of $(2l+1) \times (2l+1)$ matrices $\{U^l \,|\, l = 0, 1, 2, ...\}$ are called irreducible unitary representations (IURs) which provides the basis of the Fourier transform for a Lie group (see [32], [41] for the detailed explanation). They have the following important properties [32], [42]

$$U^l(R_1R_2) = U^l(R_1)U^l(R_2) \;\text{ and }\; U^l(R^{-1}) = (U^l(R))^* \tag{8}$$

where $*$ denotes the Hermitian conjugate. Explicitly, these matrices are given in terms of functions of mathematical physics such as the Jacobi polynomials. These matrix functions are smooth functions of R in the sense that

$$u^l(X) \doteq \frac{d}{dt} U^l(\exp tX)\Big|_{t=0} \tag{9}$$

exists, as do all higher derivatives. These matrices are representations of the Lie algebra $so(3)$, and are linear in their argument:

$$u^l(X) \;=\; \sum_{i=1}^{3} x_i u^l(E_i). \tag{10}$$

The function $f(R)$ is recovered from the set of coefficients $\{\hat{f}^l_{nm}\,|\,l=0,1,2,...\}$ with the Fourier series

$$f(R) \;=\; \sum_{l=0}^{\infty}(2l+1)\sum_{m,n=-l}^{l}\hat{f}^l_{nm}U^l_{mn}(R) \tag{11}$$

Fourier analysis is a natural tool for propagation of uncertainty because of the convolution theorem, which gives

$$(\widehat{f_t * f_{\Delta t}})^l_{mn} = \sum_{k=-l}^{l}(\widehat{f_{\Delta t}})^l_{mk}(\widehat{f_t})^l_{kn}. \tag{12}$$

Note the reversal of order of the products, which is significant in this noncommutative setting.

In the Fourier setting, a concentrated Gaussian distribution defined in (1) can be computed with a bandlimited series wherein l is truncated at a high number. The Fourier coefficients for this series can be computed as

$$\begin{aligned}
(\widehat{f_\Sigma})^l &= \int_{SO(3)} f_\Sigma(R)U^l(R^{-1})dR\\
&= C\int_{\mathbb{R}^3} e^{-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}}\\
&\cdot\left(\mathbb{I}_{2l+1}+u^l(-X)+\frac{1}{2}(u^l(-X))^2+\cdots\right)|J(\mathbf{x})|d\mathbf{x}\\
&= C\int_{\mathbb{R}^3} e^{-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}}\\
&\cdot\left(\mathbb{I}_{2l+1}+\frac{1}{2}\sum_{i,j=1}^{3}x_ix_ju^l(E_i)u^l(E_j)+\cdots\right)|J(\mathbf{x})|d\mathbf{x}\\
&= \exp\left(\frac{1}{2}\sum_{i,j=1}^{3}\sigma_{ij}u^l(E_i)u^l(E_j)\right).
\end{aligned}$$

Here $u^l(\cdot)$ was defined in (9) and the property (10) was used. The constant C can be computed as $\frac{1}{(2\pi)^{\frac{3}{2}}|\Sigma|^{\frac{1}{2}}}$ in the concentrated Gaussian cases. Though our arguments here are approximations, it can be shown that the final result given above is exact [32], [41].

Similarly, if a system is governed by (4), then the Fourier transform of the pdf describing an infinite ensemble is

$$\begin{aligned}
(\widehat{f_t})^l = \exp\Big(t\cdot\sum_{k=1}^{3}a_k\,u^l(E_k)\\
+\frac{1}{2}t\cdot\sum_{i,j=1}^{3}(BB^T)_{ij}u^l(E_i)u^l(E_j)\Big).
\end{aligned} \tag{13}$$

And this too is exact. When the quantities in the matrix exponential are small, then the first two terms in the Taylor series can be used.

IV. Bayesian Fusion Using Fourier Analysis

A. Case I: $SO(2)$

Let us first, as a motivation example, consider two functions on the circle $f(\theta)$ and $g(\theta)$, where the variable to describe the functions is $\theta\in\mathbb{S}^1$. We want to compute $f(\theta)\,g(\theta)$ using Fourier approach. First, let

$$f(\theta) = \sum_{n=-\infty}^{\infty}\hat{f}_n\,e^{\mathrm{i}\,n\,\theta}$$

and

$$g(\theta) = \sum_{m=-\infty}^{\infty}\hat{g}_m\,e^{\mathrm{i}\,m\,\theta}$$

where

$$\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi}f(\theta')\,e^{-\mathrm{i}\,n\,\theta'}\,d\theta'$$

and similarly for $\hat{g}_m$. The convolution theorem gives $\widehat{(f*g)}_k = \hat{f}_k\,\hat{g}_k$ where

$$(f*g)(\theta) = \int_0^{2\pi}f(\xi)\,g(\theta-\xi)\,d\xi.$$

In contrast, if we want to fuse two pdf's on $SO(2)\cong\mathbb{S}^1=\mathbb{T}^1$, then we seek to express the product $f(\theta)\cdot g(\theta)$ in a Fourier series. The Fourier coefficients of this product will be

$$\widehat{(f\cdot g)}_k = \sum_{m,n=-\infty}^{\infty}\hat{f}_n\hat{g}_m\,\frac{1}{2\pi}\int_0^{2\pi}e^{\mathrm{i}n\theta}\,e^{\mathrm{i}m\theta}\,e^{-\mathrm{i}k\theta}\,d\theta$$

where the last integral term becomes $\delta_{n+m,k}$. Hence

$$\widehat{(f\cdot g)}_k = \sum_{n=-\infty}^{\infty}\hat{f}_n\,\hat{g}_{n-k} = \left(\hat{f}\star\hat{g}\right)_k \tag{14}$$

where $\star$ denote the convolution of discrete signals. If f and g are band limited with band limit N (so that $-N\le m,n\le N$) and if $2N+1\le 2^B$, then (14) can be computed by FFT in $O(B\log_2 B)$ time.

B. Case II: $SO(3)$

Now let us move the discussion onto the functions on $SO(3)$. Denote the prior as $f(R)$ and the measurement distribution as $g(R)\doteq f(z\,|\,R)$. Moreover, suppose that both of these distributions are bandlimited Fourier expansions on $SO(3)$ of the form

$$\begin{aligned}
f(R) &= \sum_{l_1=0}^{N}(2l_1+1)\sum_{m_1,n_1=-l_1}^{l_1}\hat{f}^{l_1}_{n_1,m_1}U^{l_1}_{m_1,n_1}(R)\\
g(R) &= \sum_{l_2=0}^{N}(2l_2+1)\sum_{m_2,n_2=-l_2}^{l_2}\hat{g}^{l_2}_{n_2,m_2}U^{l_2}_{m_2,n_2}(R).
\end{aligned} \tag{15}$$

Note that we use the same band-limit for both $f(R)$ and $g(R)$.

We seek the bandlimited approximation of the posterior distribution given by Bayes' rule

$$f(R\,|\,z) \;=\; \frac{f(z\,|\,R)f(R)}{\int_{SO(3)}f(z\,|\,R)f(R)dR} \tag{16}$$

First, let us consider the numerator. By using the Fourier expansions of each function, we obtain

$$f(R)\,g(R) = \sum_{l_1=0}^{N}\sum_{l_2=0}^{N}(2l_1+1)\,(2l_2+1) \cdot \sum_{m_1,n_1=-l_1}^{l_1}\;\sum_{m_2,n_2=-l_2}^{l_2}\hat{f}^{l_1}_{n_1,m_1}U^{l_1}_{m_1,n_1}(R)\,\hat{g}^{l_2}_{n_2,m_2}U^{l_2}_{m_2,n_2}(R) \tag{17}$$

Now we want to compute the Fourier transform of the product of two density functions, which is computed as

$$\widehat{(f\,g)}^{l}_{n,m} = \int_{SO(3)} f(R)\,g(R)\,\overline{U^{l}_{m,n}(R)}\,dR. \tag{18}$$

If we substitute (17) into the above equation, then it follows

$$\widehat{(f\,g)}^{l}_{n,m} = \sum_{l_1,l_2=0}^{N}(2l_1+1)\,(2l_2+1)\sum_{m_1,n_1,m_2,n_2}\hat{f}^{l_1}_{n_1,m_1}\hat{g}^{l_2}_{n_2,m_2} \cdot \int_{SO(3)}U^{l_1}_{m_1,n_1}(R)\,U^{l_2}_{m_2,n_2}(R)\,\overline{U^{l}_{m,n}(R)}\,dR. \tag{19}$$

The integral term can be expressed with Clebsch Gordan coefficients, denoted as $C^{l,m}_{l_1,m_1;l_2,m_2}$, which are widely used in particle physics (see [42] for the definition), as [41]

$$\begin{aligned}&\int_{SO(3)}U^{l_1}_{m_1,n_1}(R)\,U^{l_2}_{m_2,n_2}(R)\,\overline{U^{l}_{m,n}(R)}\,dR\\ &= \frac{1}{2l+1}\,C^{l,m}_{l_1,m_1;l_2,m_2}\,C^{l,n}_{l_1,n_1;l_2,n_2}\end{aligned} \tag{20}$$

hence we obtain

$$\widehat{(f\,g)}^{l}_{n,m} = \sum_{l_1=0}^{N}\sum_{l_2=0}^{N}\frac{(2l_1+1)(2l_2+1)}{2l+1} \sum_{m_1,n_1=-l_1}^{l_1}\;\sum_{m_2,n_2=-l_2}^{l_2}\hat{f}^{l_1}_{n_1,m_1}\,\hat{g}^{l_2}_{n_2,m_2}\,C^{l,m}_{l_1,m_1;l_2,m_2}\,C^{l,n}_{l_1,n_1;l_2,n_2}. \tag{21}$$

Furthermore, noting the condition that makes Clebsch Gordan coefficients non-zero, i.e.,

$$\begin{aligned}&C^{l,m}_{l_1,m_1;l_2,m_2}\,C^{l,n}_{l_1,n_1;l_2,n_2} =\\ &\delta_{m,m_1+m_2}\,\delta_{n,n_1+n_2}\,C^{l,m}_{l_1,m_1;l_2,m_2}\,C^{l,n}_{l_1,n_1;l_2,n_2},\end{aligned}$$

it can be further simplified to

$$\widehat{(f\,g)}^{l}_{n,m} = \sum_{l_1=0}^{N}\sum_{l_2=0}^{N}\frac{(2l_1+1)(2l_2+1)}{2l+1} \sum_{m_1,n_1=-l_1}^{l_1}\hat{f}^{l_1}_{n_1,m_1}\,\hat{g}^{l_2}_{n-n_1,m-m_1}\,C^{l,m}_{l_1,m_1;l_2,m-m_1}\,C^{l,n}_{l_1,n_1;l_2,n-n_1} \tag{22}$$

where $-l_2 \le n-n_1, m-m_1 \le l_2$ to make $\hat{g}^{l_2}_{n-n_1,m-m_1}$ nonzero.

Regarding the denominator, first noting that [42]

$$U^{l_2}_{m_2,n_2} = (-1)^{n_2-m_2}\,\overline{U^{l_2}_{-m_2,-n_2}} \tag{23}$$

it follows that

$$\begin{aligned}&\int_{SO(3)} f(R)\,g(R)\,dR =\\ &\sum_{l_1,l_2=0}^{N}(2l_1+1)(2l_2+1)\sum_{m_1,n_1,m_2,n_2}\hat{f}^{l_1}_{n_1,m_1}\,\hat{g}^{l_2}_{n_2,m_2}\\ &(-1)^{n_2-m_2}\int_{SO(3)}U^{l_1}_{m_1,n_1}(R)\,\overline{U^{l_2}_{-m_2,-n_2}(R)}\,dR\\ &= \sum_{l_1=0}^{N}(2l_1+1)\sum_{m_1,n_1=-l_1}^{l_1}(-1)^{m_1-n_1}\,\hat{f}^{l_1}_{n_1,m_1}\,\hat{g}^{l_1}_{-n_1,-m_1}\end{aligned} \tag{24}$$

due to the orthogonality of IURs such that

$$\begin{aligned}&\int_{SO(3)}U^{l_1}_{m_1,n_1}(R)\,\overline{U^{l_2}_{m_2,n_2}(R)}\,dR\\ &= \frac{1}{2l_1+1}\delta_{l_1,\,l_2}\,\delta_{m_1,\,m_2}\,\delta_{n_1,\,n_2}.\end{aligned} \tag{25}$$

Note that in computing volume of the function on $SO(3)$ in the current case, one needs to include volume of $SO(3)$ which is $8\pi^2$. In other words, the denominator should be multiplied by $8\pi^2$ for the correct normalization.

Therefore, with the same band limit N, the posterior distribution function can be obtained via Fourier transforms of the prior and the measurement distributions.

In general this will have a much higher bandlimit than the original functions. But by computing its Fourier coefficients and truncating at the same bandlimit, the most essential information can be retained. This, in particular, is suitable for the cases of large variance as shown in figure 2.

V. Numerical Demonstration

A. $SO(2)$ case

Let us try the following two Gaussians

$$\begin{aligned}f(\theta) &= \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma_1^2}\right)\\ g(\theta) &= \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma_2^2}\right)\end{aligned}$$

where $\theta \in [-\pi,\,\pi]$. Figure 1 shows the comparison between the original and reconstruction of the product of $f(\theta)$ and $g(\theta)$. Here we choose $\sigma_1 = 1.1$ and $\sigma_2 = 1.3$ which is the case of truncated Gaussians. Also, we consider the Gaussians of the same mean values in this and the next section, although the cases of different means can be treated without loss of generality by shifting the mean to non-identity values, as in Section II. Fourier transform approach used $N = 2^3$, which is small but already good enough to reconstruct $f(\theta)\cdot g(\theta)$. As is well-known, FFT approach becomes more efficient when N becomes large, which could be shown from the current example (average time for brute force computation: 0.006 sec,

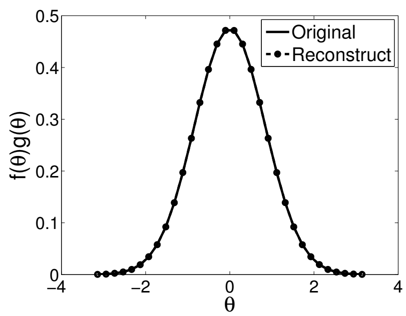

Fig. 1. Product of two Gaussian functions ($\sigma_1 = 1.1$ and $\sigma_2 = 1.3$ are used). Lower value of the bandlimit such as $N = 2^3$ is already good for matching reconstructed function with the original one.

whereas using FFT: 3.3×10^{-5} sec, running from Matlab R2015a on the computer CPU 1.8 GHz, Intel Core i5, OS X 10.9.4).

B. $SO(3)$ case

Given a rotation matrix $R = \exp(X)$, the corresponding Lie algebra elements are simply $\log^\vee(R) = X = \sum_{i=1}^{3} x_i\, E_i$ where $\mathbf{x} = [x_1\ x_2\ x_3]^T \in B_\pi$. With the above information, one can define Gaussian functions with each mean being identity as examples of $f(R)$ and $g(R)$, as

$$f(R) = C_1 \exp\left(-\frac{1}{2}[\log^\vee(R)]^T \Sigma_1^{-1} \log^\vee(R)\right) \tag{26}$$

and

$$g(R) = C_2 \exp\left(-\frac{1}{2}[\log^\vee(R)]^T \Sigma_2^{-1} \log^\vee(R)\right). \tag{27}$$

When the Gaussians are concentrated, then $C_i = \frac{1}{(2\pi)^{3/2}|\Sigma_i|^{1/2}}$ $(i = 1, 2)$, whereas when Gaussians are widely spread, then C_i can be computed using integrating $\exp\left(-\frac{1}{2}[\log^\vee(R)]^T \Sigma_i^{-1} \log^\vee(R)\right)$. Here we assume that $\Sigma_1 = \mathrm{diag}(0.28, 0.55, 0.55)$ and $\Sigma_2 = \mathrm{diag}(0.40, 0.63, 0.63)$. Fourier transform of each function can be computed by using Section III. IURs for the reconstruction can be computed as

$$U^l(R) = U^l(\exp(X)) = \exp\left(\sum_{i=1}^{3} x_i\, u^l(E_i)\right).$$

Figure 2 shows the comparison between the original form of the product of two functions (normalized so that they are pdf's) as in (26) and (27) (Fig. 2 A,C,E) and the reconstruction from Fourier filtering approach (Fig. 2 B,D,F), at different values of x_3. This result shows that the Fourier filtering approach on $SO(3)$ retains the information on the original functions well enough.

Under the assumption of smallness, $R = \exp(X) \approx \mathbb{I} + X$ and similarly from (13)

$$(\widehat{f_{\Delta t}})^l \approx \mathbb{I}_{2l+1} + \Delta t \cdot \Big(\sum_{k=1}^{3} a_k\, u^l(E_k) + \frac{1}{2}\sum_{i,j=1}^{3} (BB^T)_{ij} u^l(E_i) u^l(E_j)\Big).$$

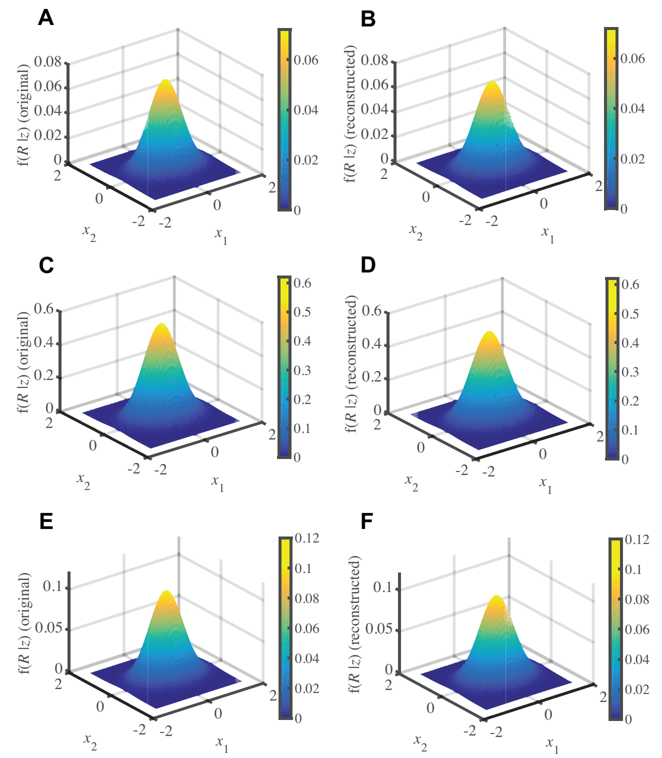

Fig. 2. Normalized product of two Gaussian functions (Σ_i's are in the main text) at different x_3 values. (A,C,E) represent the original function plots, whereas (B,D,F) are reconstructed ones. (A,B) when $x_3 = -1.10$; (C,D) when $x_3 = -0.052$; (E,F) when $x_3 = 0.99$. Lower value of the bandlimit such as $N = 7$ is good enough for matching reconstructed function with the original one.

An analogous expression holds for $(\widehat{f_\Sigma})^l$ when $\|\Sigma\|$ is small. This is significant from a computational perspective because the matrices $u^l(E_i)$ are tri-diagonal, which means that the cost of matrix multiplication in the convolution theorem (during the propagation step) and the evaluation of summations during the Bayesian fusion step can be computed more efficiently than in the case when the distributions are more spread out. On the other hand, the value of l must be taken up to a high level in this case.

When the underlying distributions have high covariance, all Fourier matrices are full matrices without an advantageous zero structure. However, in this case bandlimited approximations with small values of l can be used.

VI. Conclusions

We present a filtering method on the rotation group, $SO(2)$ and $SO(3)$. Both the propagation of uncertainty and the fusion of priors with measurement distributions are performed using Fourier analysis on the rotation group. Though Gaussian methods in exponential coordinates are far faster than Fourier methods when the probability density functions in the filter are highly concentrated, those methods break down for large covariances. In contrast, Fourier based filters such as those presented here handle the opposite extreme when distributions are very spread out, and where the very definition of covariance breaks down. In this case, the important issue here is about the accuracy, rather than about computation speed, since the tails of the distribution might not decay inside the ball of radius π. Therefore, Fourier-based filtering on groups opens up opportunities for the very high uncertainty case.

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