# Principal Kinematic Inequalities

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**Abstract.** One hundred years ago, Blaschke and Poincaré derived exact closed-form formulas for integrals over $SE(n)$ of Euler characteristics of intersections of $n$-dimensional bodies for $n = 2, 3$. Three decades thereafter Chern extended these formulas to the $n$-dimensional case. These results form a core tool in the fields of convex geometry, integral geometry, geometric probability, and stochastic geometry. These results, often referred to as "Principal Kinematic Formulae," are extended here to the case of integrals of set-indicator functions, resulting in inequalities for the case of non-convex bodies. These results are relevant to assessing the frequency of occurrence of collisions that occur in sample-based robot motion planning.

**Keywords:** integral geometry, Brunn-Minkowski, kinematic formula, convex bodies, inequalities

## 1 Introduction and Literature Review

As is well known in Kinematics, $SE(n)$ is the group of rigid body motions with group law [1, 6, 16, 20, 23, 31]

$$(R_1, \mathbf{t}_1) \circ (R_2, \mathbf{t}_2) = (R_1 R_2, R_1 \mathbf{t}_2 + \mathbf{t}_1)$$

where $R_i \in SO(n)$, the special orthogonal group consisting of $n \times n$ rotation matrices. The above group law makes $SE(n)$ a semi-direct product,

$$SE(n) = \mathbb{R}^n \rtimes SO(n)$$

with a natural bi-invariant volume element with which to integrate [9, 10], despite the fact that $SE(n)$ does not have a bi-invariant metric. This paper makes extensive use of the bi-invariant integration measure for $SE(n)$ to compute bounds on the volume of the part of $SE(n)$ that puts moving bodies in collision. The derived results hold even when these bodies are not convex.

Let $B_i$ be a finite solid body in $\mathbb{R}^n$ with closed surface $\partial B_i$ for $i \in \{0, 1\}$. This paper is concerned with properties of the intersection of two such bodies. In particular, suppose that $B_0$ is held fixed, and for an arbitrary rigid-body motion $g = (R, \mathbf{t}) \in SE(n)$,

$$g \cdot B_1 \doteq \{R\mathbf{x} + \mathbf{t} \,|\, \mathbf{x} \in B_1\}$$

is a rigidly moved copy of $B_1$.

It is then possible to define functions of motion such as

$$p(g) \doteq \phi(B_0 \cap g \cdot B_1) \text{ and } s(g) \doteq \psi(\partial B_0 \cap g \cdot \partial B_1).$$

For example, for each fixed value of $g \in SE(n)$, $\phi$ might be the Euler characteristic, volume, or indicator function of the intersection body, and $\psi$ might be the Euler characteristic, area, or mean curvature of the intersection surface.

A major part of the field of integral geometry/geometric probability is concerned with computing closed-form expressions for integrals of functions such as $p(g)$ and $s(g)$. One of the primary results from this field is that if $\phi$ is the Euler characteristic, then a closed-form expression results for the integral

$$\Xi(B_0, B_1) \doteq \int_{SE(n)} \chi(B_0 \cap g \cdot B_1)\, dg$$

where $\chi$ is the Euler charactersitic and $dg$ is the Haar measure for the group $SE(n)$. This $dg$ is the unique measure (up to scaling) that has the property that for all $f \in L^1(SE(3))$

$$\int_{SE(3)} f(g)\, dg = \int_{SE(3)} f(g^{-1})\, dg = \int_{SE(3)} f(h \circ g)\, dg = \int_{SE(3)} f(g \circ h)\, dg \quad (1)$$

for arbitrary $h \in SE(3)$. In particular, for rigid-body motions in the plane parameterized in terms of translations expressed in Cartesian coordinates $(x, y)$ and rotation angle $\theta$, $dg = dxdyd\theta$. For motions in 3D with translations expressed in Cartesian coordinates $(x, y, z)$ and rotations expressed in terms of ZXZ Euler angles $(\alpha, \beta, \gamma)$, $dg = \sin\beta\, d\alpha d\beta d\gamma dx dy dz$. More generally, the procedure for constructing the expression for $dg$ in any dimension and in any parameterization is outlined in [9, 10].

Remarkably, the closed form integral for $\Xi(B_0, B_1)$ can be obtained for arbitrary compact bodies (not only convex ones) with continuous piecewise differentiable boundaries, and can be expressed in the form [4, 5, 7, 18, 24, 28, 29]

$$\Xi(B_0, B_1) = \sum_{i=0}^{n} c_i P_i(B_0) P_{n-i}(B_1) \quad (2)$$

where each $c_i$ is a known constant and each $P_i(B_k)$ is an integral of the body (or its bounding surface), such as the Euler characteristic, volume, surface area, integral of mean curvature, etc. Equation (2) is called the Principal Kinematic Formula.

Due to Hadwiger's characterization theorem [14, 15], it is also possible to compute closed form integrals when $\phi$ is taken to be the volume of intersection, or $\psi$ is taken to be any of the other $P_i$ functions (which are called mixed-volumes in Brunn-Minkowski theory, or quermassintegrals in Geometric Probability).

The following subsections review this formula in the cases of planar and spatial bodies.

### 1.1 The 2D Case for Convex Bodies

Let $C_0$ and $C_1$ be two compact convex planar bodies with corresponding continuous piecewise differentiable boundaries $\partial C_0$ and $\partial C_1$. Let $A(C_i)$ denote the area of $C_i$ and let $L(C_i)$ denote its perimeter, i.e., the length of the curve $\partial C_i$. For arbitrary bodies $B_0$ and $B_1$ use the same definitions where $L$ counts in addition the total length of any internal boundaries for a non-simply connected body. Then the following theorem holds.

**Theorem 1.** *(Blaschke, [4, 5]): Given arbitrary compact planar bodies $B_0$ and $B_1$ with continuous piecewise differentiable boundaries, then (2) evaluates as*

$$\varXi(B_0, B_1) = 2\pi[A(B_0)\chi(B_1) + A(B_1)\chi(B_0)] + L(B_0)L(B_1)\,. \tag{3}$$

*and for planar compact convex bodies $C_0$ and $C_1$, then (2) evaluates as*

$$\varXi(C_0, C_1) = 2\pi[A(C_0) + A(C_1)] + L(C_0)L(C_1)\,. \tag{4}$$

For the proof see [4, 5, 18, 28].

### 1.2 The 3D Case for Convex Bodies

Let $C_i$ for $i \in \{0, 1\}$ be compact convex bodies in $\mathbb{R}^3$. If the spatial body $C_i$ has a continuous piecewise differentiable surface, $\partial C_i$, that we can compute the total surface area

$$\int_{\partial C_i} dS = F(C_i)$$

Furthermore, if $\kappa$ denotes the Gaussian curvature at each point on the surface, we can compute (via the Gauss-Bonnet Theorem):

$$\int_{\partial C_i} \kappa\, dS = 2\pi\, \chi(\partial C_i)$$

where $\chi(\partial C_i)$ is the Euler characteristic of the bounding surface. The Euler characteristic of a body and the surface that bounds it are related as

$$\chi(\partial C_i) = 2 \cdot \chi(C_i).$$

Moreover, for simply connected planar and spatial bodies $\chi(C_i) = \iota(C_i) = 1$ where the *indicator function* on any measurable body, $B$, (not necessarily convex and perhaps not even connected) is defined by:

$$\iota(B) \doteq \begin{cases} 1 \ \text{ if } B \neq \emptyset \\ 0 \text{ for } B = \emptyset \end{cases}$$

In differential geometry, $\kappa$ is not the only kind of curvature. A second kind of curvature is defined at every point on a surface. This is the *mean curvature*, $m$. The total mean sectional curvature is defined as

$$M(C_i) = \int_{\partial C_i} m\, dS.$$

Let $V(C_i)$ denote the volume of $C_i$. Then we have the following theorem.

The evaluation of $M$ and $F$ for nonconvex simply connected bodies follows in the same way, and for non-simply-connected bodies it follows by extending the computations to the surfaces of any internal voids.

**Theorem 2.** *(Blaschke/Poincaré, [4, 5, 24]): Given arbitrary compact 3D bodies $B_0$ and $B_1$ with continuous piecewise differentiable boundaries, then (2) evaluates as*

$$\Xi(B_0, B_1) = 8\pi^2[V(B_0)\chi(B_1)+V(B_1)\chi(B_0)]+2\pi[F(B_0)M(B_1)+F(B_1)M(B_0)]\,. \tag{5}$$

*Given compact convex bodies $C_0$ and $C_1$ in $\mathbb{R}^3$, then (2) evaluates as*

$$\Xi(C_0, C_1) = 8\pi^2[V(C_0) + V(C_1)] + 2\pi[F(C_0)M(C_1) + F(C_1)M(C_0)]\,. \tag{6}$$

For the proof see [4, 5, 18, 28].

### 1.3 Main Goal of This Paper

In the context of robot motion planning algorithms, one would like to assess the amount of free space that a body has to move without colliding with other bodies. As such, it becomes useful to evaluate

$$\mathcal{I}(B_0, B_1) \doteq \int_{SE(n)} \iota(B_0 \cap g \cdot B_1)\, dg\,.$$

Moreover, if the intersection body $B_0 \cap g \cdot B_1$ is convex, then $\iota(B_0 \cap g \cdot B_1) = \chi(B_0 \cap g \cdot B_1)$. If this holds for all motions $g \in SE(n)$ which contribute to the integrals, then the integrals themselves will be equal. In particular, since intersections of convex bodies are always either convex or empty, given compact convex bodies $C_0$ and $C_1$ we can say

$$\mathcal{I}(C_0, C_1) = \Xi(C_0, C_1),$$

which can be evaluated using the Principal Kinematic Formula (2).

However, in general since for nonconvex bodies in $\mathbb{R}^n$, it can happen that intersections can either have multiple components (which have $\chi > 1$) and toroidal regions (which have $\chi < 1$), these both differ from the value of $\iota = 1$ on nonempty intersections. Hence, for nonconvex bodies in general

$$\mathcal{I}(B_0, B_1) \neq \Xi(B_0, B_1).$$

The main goal of this paper is therefore to establish bounds of the form

$$\Xi(B_0^L, B_1^L) \leq \mathcal{I}(B_0, B_1) \leq \Xi(B_0^U, B_1^U) \tag{7}$$

which justifies the title of this paper.

### 1.4 Literature Review

The topics of convex and integral geometry have a long and distinguished history, often intertwining with kinematics. Classical works in this area include [8, 13–15, 17, 24, 25]. Comprehensive yet accessible monographs introducing this topic include [26, 28, 29]. Relatively recent expositions of these topics include [18, 19]. Extensions of the principal kinematic formula include [21, 27].

Ideas related to this formula have been explored previously by the author in [12, 32], which provide motivations for the current presentation.

## 2 Integral Geometry of the Indicator Function

In this section, the properties of the Haar measure for $SE(3)$ are combined with properties of the indicator function to produce the desired inequalities.

In combination with the invariance of the integral in (1), we find that

$$\mathcal{I}(C_0, C_1) \,=\, \int_{SE(n)} \iota(C_0 \cap g^{-1} \cdot C_1)\, dg \,=\, \int_{SE(n)} \iota(C_1 \cap g \cdot C_0)\, dg \,=\, \mathcal{I}(C_1, C_0) \tag{8}$$

Due to the semi-direct-product structure of $SE(n)$, the integral of any integrable function $f : SE(n) \to \mathbb{R}$ can be decomposed as

$$\int_{SE(n)} f(g)\, dg = \int_{\mathbb{R}^n} \int_{SO(n)} f(R, \mathbf{t})\, dR\, d\mathbf{t} = \int_{SO(n)} \int_{\mathbb{R}^n} f(R, \mathbf{t})\, d\mathbf{t}\, dR\,.$$

Applying this principle, we find that

$$\mathcal{I}(B_0, B_1) \,=\, \int_{SO(n} \int_{\mathbb{R}^n} \iota(B_0 \cap (R \cdot B_1 + \mathbf{t}))\, d\mathbf{t}\, dR.$$

If $R \cdot B_1$ is denoted as $B_2$, note that for each fixed $R$,

$$\int_{\mathbb{R}^n} \iota(B_0 \cap (B_2 + \mathbf{t}))\, d\mathbf{t} = V(B_0 \oplus (-B_2))$$

where $\oplus$ denotes the Minkowski sum, and $-B_i$ denotes the centrally inverted version of $B_i$ which maps each point $\mathbf{x} \in B_i$ to $-\mathbf{x}$. Since $-I$ and $R \in SO(3)$ commute,

$$\mathcal{I}(B_0, B_1) \,=\, \int_{SO(n)} V(B_0 \oplus (-R \cdot B_1))\, dR. \tag{9}$$

## 3 Principal Kinematic Inequalities

Other than for the case of convex bodies, general equalities for $\mathcal{I}(B_0, B_1)$ have not been derived over the past 100 years of integral geometry. Here several inequalities are derived.

### 3.1 Inscribing and Circumscribing Convex Bodies

Suppose that we have two not-necessarily-convex bodies, $B_0$ and $B_1$. Let $B_0$ be stationary, and let $B_1$ be mobile. If it is possible to inscribe a convex body $C_i^L \subseteq B_i$ for $i \in \{0,1\}$, and to enclose $B_i$ with a convex body $C_i^U$ (e.g., the convex hull), then obviously (7) can be achieved as

$$\Xi(C_0^L, C_1^L) \leq \mathcal{I}(B_0, B_1) \leq \Xi(C_0^U, C_1^U)\,.$$

### 3.2 Using the Brunn-Minkowski Inequality

Building on (9), and raising the Brunn-Minkowski inequality

$$V(B_0 \oplus B_2)^{1/n} \geq V(B_0)^{1/n} + V(B_2)^{1/n}$$

to the $n^{th}$ power, and observing that $V(-R \cdot B_1) = V(B_1)$ gives

$$\mathcal{I}(B_0, B_1) \geq Vol(SO(n)) \sum_{k=0}^{n} \binom{n}{k} |V(B_0)|^{k/n} |V(B_1)|^{(n-k)/n}. \tag{10}$$

Note that $Vol(SO(2)) = 2\pi$ and $Vol(SO(3)) = 8\pi^2$. Similarly, an upper-bound can be obtained from the reverse Brunn-Minkowski inequality.

### 3.3 Special Non-Convex Cases when $\mathcal{I}(B_0, B_1) \leq \Xi(B_0, B_1)$

In the plane, if $B_0$ and $B_1$ are simply connected (but not necessarily convex), then there is no way to generate an annular region by performing intersections of the form $B_0 \cap g \cdot B_1$, and hence

$$\iota(B_0 \cap g \cdot B_1) \leq \chi(B_0 \cap g \cdot B_1).$$

This is not always true in 3D because, for example, two cup-like objects can intersect at their lips producing a solid torus with $\chi = 0$ rather than $\iota = 1$. And the same can happen when intersecting a cup-shaped object and a convex object. However, two 3D non-convex spire-shaped objects (such as the volume of revolution generated from a nonconvex planar curve) will never produce such tori, nor will the intersection of a spire and a convex body. And so, in these cases $\mathcal{I}(B_0, B_1) \leq \Xi(B_0, B_1)$ as well.

### 3.4 Special Nonconvex Cases When $\mathcal{I}(B_0, B_1) = \Xi(B_0, B_1)$

In every dimension, if all nonempty intersections of $B_0$ and $g \cdot B_1$ always have a single component, then $\iota(B_0 \cap g \cdot B_1) = \chi(B_0 \cap g \cdot B_1)$, regardless of whether or not $B_0$ and $B_1$ are convex. Consequently, in such cases $\mathcal{I}(B_0, B_1) = \Xi(B_0, B_1)$.

Such examples can be constructed. Let $D$ be a solid disk/ball and let $C_0$ be a convex body. Choose a translation $\mathbf{t} \in \partial C_0$ and define $B_0 \doteq C_0 \cap (\mathbf{t} + D)$. If $D$ is chosen to be small enough, then $B_0$ can be guaranteed to be simply connected. In this case, given a second disk/ball $D' \subset D$, we find that $B_0 \cap g \cdot D'$ has a single component, and so $\mathcal{I}(B_0, D') = \Xi(B_0, D')$. Moreover, if $C_1$ is a convex body such that $C_1 \subset D'$ and the curvatures at every point on $\partial C_1$ are greater than those of $D'$, then it is guaranteed that $\iota((C_0 \cap (\mathbf{t} + D)) \cap g \cdot C_1) = \chi((C_0 \cap (\mathbf{t} + D)) \cap g \cdot C_1)$, and hence

$$\mathcal{I}(C_0 \cap (\mathbf{t} + D), C_1) = \Xi(C_0 \cap (\mathbf{t} + D), C_1) . \tag{11}$$

Relaxing the curvature conditions results in the upper bound $\mathcal{I} \leq \Xi$.

## 4 Conclusions

In the fields of integral and convex geometry, the principal kinematic formula plays a central role. Two versions of this formula typically appear: (1) closed-form expressions for the integral over motion of the Euler characteristic of nonconvex bodies; (2) closed-form expressions for the integral over motion of any quermassintegral/mixed-volume of convex bodies. In both cases, Hadwiger's characterization theorem applies. However, in robot motion planning one is concerned with determining which regions of the configuration space of a rigid body correspond to collision. Such collisions are detected by evaluating the set-indicator function of potentially non-convex moving bodies with stationary obstacles. Integral and convex geometry do not address this directly, and so inequalities are derived and presented to assess the integral over motion of the indicator function of intersections of bodies.

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